Lecture 9  Forward Risk Adjusted Probability Measures and Fixed-income Derivatives

9.1 Forward risk adjusted probability measures

This section is a preparation for valuation of fixed-income derivatives. For a \( \mathcal{F}_\tau \)-measurable contingent claim \( Y(\tau) \), its time \( t \) value is given by

\[
V(t) = \mathbb{E}_Q[Y(\tau)B(t)/B(\tau) \mid \mathcal{F}_t]
\]

(9.1)

with a risk neutral probability measure \( Q \). (9.1) is particularly convenient when the bank account process \( B \) is deterministic, because it simply becomes

\[
V(t) = B(t)/B(\tau) \mathbb{E}_Q[Y(\tau) \mid \mathcal{F}_t].
\]

(9.2)

In the case of stochastic interest rate \( r \), an alternative method is developed based on the following change of measures.

For a fixed \( \tau \leq T \), let \( M(\tau) > 0 \) be a \( \mathcal{F}_\tau \)-measurable random variable satisfying \( \mathbb{E}_Q M(\tau) = 1 \). Define a new probability measure \( P^\tau \) by

\[
P^\tau(\omega) = M(\tau; \omega) Q(\omega) \quad \forall \omega \in \Omega.
\]

(9.3)

Obviously \( P^\tau \) is a probability measure and \( P^\tau(\omega) > 0 \) for all \( \omega \). Let \( E^\tau(\cdot) \) denote the expectation corresponding to \( P^\tau \). Define a \( Q \)-martingale \( M = \{M(t) : t = 0, 1, \ldots, \tau\} \) by

\[
M(t) = \mathbb{E}_Q[M(\tau) \mid \mathcal{F}_t].
\]

Now we state a basic result of changing martingale measures.

**Proposition 9.1** \( MY = \{M(t)Y(t) : t = 0, 1, \ldots, \tau\} \) is a \( Q \)-martingale if and only if \( Y = \{Y(t) : t = 0, 1, \ldots, \tau\} \) is a \( P^\tau \)-martingale.

**Proof** A major step is to show that

\[
E^\tau[M(t)Y(\tau) \mid \mathcal{F}_t] = \mathbb{E}_Q[M(\tau)Y(\tau) \mid \mathcal{F}_t] \quad \forall t \leq \tau.
\]

(9.4)

To verify (9.4), take an arbitrary event \( A \in \mathcal{F}_t \). Then \( M(t) \) is constant on \( A \) and

\[
M(t; \omega) = \mathbb{E}_Q[M(\tau)|A] = \sum_{\omega' \in A} M(\tau; \omega') Q(\omega')/Q(A) = P^\tau(A)/Q(A) \quad \forall \omega \in A.
\]
Therefore,

\[
E^\tau[M(t)Y(\tau) \mid A] = M(t) E^\tau[Y(\tau) \mid A]
= P^\tau(A)/Q(A) \sum_{\omega \in A} Y(\tau; \omega)M(\tau; \omega)Q(\omega)/P^\tau(A)
= \sum_{\omega \in A} Y(\tau; \omega)M(\tau; \omega)Q(\omega)/Q(A)
= E_Q[M(\tau)Y(\tau) \mid A].
\]

Note that \(MY\) is a \(Q\)-martingale if and only if \(M(t)Y(t) = E_Q[M(\tau)Y(\tau) \mid F_t]\) for all \(t \leq \tau\), which is equivalent to \(Y(t) = E^\tau[Y(\tau) \mid F_t]\) for all \(t \leq \tau\) by (9.4), i.e. \(Y\) is a \(P^\tau\)-martingale. This completes the proof of Proposition 9.1.

To apply Proposition 9.1 to term structure models, we first let

\[
M(\tau) = \frac{1}{B(0, \tau) B(\tau)}.
\]

(9.5)

Note that \(E_Q[M(\tau)] = \frac{E_Q[1/B(\tau)]}{B(0, \tau)} = 1\). Hence

\[
M(t) = \frac{E_Q[1/B(\tau)] \mid F_t}{B(0, \tau)} = \frac{B(t, \tau)}{B(0, \tau) B(t)}.
\]

(9.6)

Next, define the forward risk adjusted probability measure (or called the \(\tau\)-forward measure)

\[
P^\tau(\omega) = M(\tau; \omega) Q(\omega) = \frac{Q(\omega)}{B(0, \tau) B(\tau; \omega)} \quad \forall \ \omega \in \Omega.
\]

(9.7)

Let \(S(t)\) denote the time \(t\) price of a security (e.g. stock, bond, or contingent claim). Based on (6.3), define

\[
Y(t) = FO(t) = \frac{S(t)}{E_Q[B(t)/B(\tau) \mid F_t]} = \frac{S(t)}{B(t, \tau)}.
\]

(9.8)

which is the time \(t\) forward price of the security to be delivered at maturity \(\tau\). Hence

\[
M(t)Y(t) = \frac{B(t, \tau)}{B(0, \tau) B(t)} \frac{S(t)}{B(t, \tau)} = \frac{S^*(t)}{B(0, \tau)},
\]

which is the time \(t\) discounted price of the security divided by a constant. Therefore, the process \(MY\) is a \(Q\)-martingale. By Proposition 9.1, we obtain

**Proposition 9.2** Under the \(\tau\)-forward measure \(P^\tau\) defined by (9.7), the forward price process \(Y(\cdot) = FO(\cdot)\) with delivery time \(\tau\) is a martingale. Moreover,

\[
S(t) = B(t, \tau) E^\tau[S(\tau) \mid F_t].
\]

(9.9)
The formulas (9.9) and (9.1) will be used for pricing various interest rate derivatives in the next couple of sections.

**Exercises:**

9.1 In Example 7.1, calculate the forward risk adjusted measures $P^\tau$ for $\tau = 1, 2, 3$.

9.2 In the binomial tree model (Lecture 4), find the expression for $P^\tau$ with $\tau = T$, and check if (9.9) agrees with (4.2).

### 9.2 Bond options and coupon bonds

Formula (9.1) or (9.9) can be used for valuation of various interest rate derivatives.

To illustrate the computation, consider Example 7.1 again, in which $Q(\omega_i) = 1/4$ for $i = 1, \ldots, 4$, and the forward risk adjusted probability measures $P^\tau$, $\tau = 1, 2, 3$ are calculated.

<table>
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<tr>
<th>$\omega$</th>
<th>$B(0, 1)$</th>
<th>$B(0, 2)$</th>
<th>$B(1, 2)$</th>
<th>$B(0, 3)$</th>
<th>$B(1, 3)$</th>
<th>$B(2, 3)$</th>
<th>$P^1$</th>
<th>$P^2$</th>
<th>$P^3$</th>
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<td>$\omega_1$</td>
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<td>0.9203</td>
<td>0.9434</td>
<td>0.8811</td>
<td>0.8901</td>
<td>0.9346</td>
<td>0.25</td>
<td>0.2464</td>
<td>0.2451</td>
</tr>
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<td>$\sim$</td>
<td>$\sim$</td>
<td>$\sim$</td>
<td>$\sim$</td>
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<td>$\sim$</td>
<td>0.9709</td>
<td>$\sim$</td>
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<td>$\sim$</td>
<td>0.9804</td>
<td>$\sim$</td>
<td>0.2536</td>
<td>0.2597</td>
</tr>
</tbody>
</table>

Table 9.1: Data for Example 7.1 with $P^T$ added

**Example 9.1 (Bond options.)**

Consider an European call option on zero-coupon bond with the payoff $[B(\tau, s) - c]^+$. Note that $\tau$ is the maturity of the option while $s > \tau$ is the maturity of the zero-coupon bond. Set $\tau = 2$, $s = 3$ and the exercise price $c = 0.96$. Then the time $t$ price $S(t)$ with $t = 1$ is given by

\[
S(1) = B(1, 2) \cdot E^{Q}\left([B(2, 3) - 0.96]^+ \mid \mathcal{F}_1\right) \\
= B(1, 2) \cdot \left[ P^2(\omega_3 \mid \mathcal{F}_1) (0.9615 - 0.96) + P^2(\omega_4 \mid \mathcal{F}_1) (0.9804 - 0.96) \right].
\]

Hence on $\{\omega_1, \omega_2\}$, $P^2(\omega_3 \mid \mathcal{F}_1) = P^2(\omega_4 \mid \mathcal{F}_1) = 0$ and $S(1) = 0$;

while on $\{\omega_3, \omega_4\}$, $P^2(\omega_3 \mid \mathcal{F}_1) = P^2(\omega_4 \mid \mathcal{F}_1) = 1/2$ and $S(1) = 0.9709 \cdot 0.5 \cdot 0.0219 = 0.01$.

If we use (9.1) instead, then

\[
S(1) = E^{Q}\left([B(2, 3) - 0.96]^+ B(1)/B(2) \mid \mathcal{F}_1\right)
\]
\[
= [1 + r(2)]^{-1} \left[ Q(\omega_3 \mid \mathcal{F}_1) (0.9615 - 0.96) + Q(\omega_4 \mid \mathcal{F}_1) (0.9804 - 0.96) \right].
\]

Hence on \(\{\omega_1, \omega_2\}\), \(Q(\omega_3 \mid \mathcal{F}_1) = Q(\omega_4 \mid \mathcal{F}_1) = 0\) and
\[
S(1) = 0;
\]
while on \(\{\omega_3, \omega_4\}\), \(Q(\omega_3 \mid \mathcal{F}_1) = Q(\omega_4 \mid \mathcal{F}_1) = 1/2\) and
\[
S(1) = \frac{0.5 \cdot 0.0219}{1.03} = 0.01.
\]

Two methods are verified to yield the same result. In fact, Proposition 9.1 can be verified directly in this case by observing that \(B(1, 2) = B(1) \cdot B(0, 2)\).

Other bond options can be priced in the same way.

**Example 9.2** (Coupon bonds.)

A coupon bond is a contract entered at time \(t\), say, such that a sequence of coupons of \(c_1, \ldots, c_N\) dollars will be paid at times \(t < t_1 < \cdots < t_N \leq T\). By (9.1), the time \(t\) price of the coupon bond can be expressed as a linear combination of \(N\) zero-coupon bond prices:
\[
S(t) = \sum_{n=1}^{N} c_n B(t, t_n).
\] (9.10)

Now use the data in Example 9.1 and consider an European put option with maturity \(\tau = 2\), exercise price \(c = 2.25\), and based on the coupon bond with \(t_1 = 2\), \(t_2 = 3\), \(c_1 = 1.1\) and \(c_2 = 1.2\). The time \(t = 1\) price of the option is
\[
S(1) = B(1, 2) E^2\{(c - c_1 B(2, 2) - c_2 B(2, 3))^+ \mid \mathcal{F}_1\}.
\]
Hence on \(\{\omega_1, \omega_2\}\), \(c - c_1 B(2, 2) - c_2 B(2, 3) > 0\) and
\[
S(1) = 0.9434 \cdot 0.5 \cdot (2.25 - 1.1 - 1.2 \cdot 0.9346 + 2.25 - 1.1 - 1.2 \cdot 0.9524) = 0.0168;
\]
while on \(\{\omega_3, \omega_4\}\), \(c - c_1 B(2, 2) - c_2 B(2, 3) < 0\) and
\[
S(1) = 0.
\]
If we use (9.1), then
\[
S(1) = E_Q\{(c - c_1 B(2, 2) - c_2 B(2, 3))^+ B(1)/B(2) \mid \mathcal{F}_1\}.
\]
Hence on \(\{\omega_1, \omega_2\}\),
\[
S(1) = \frac{0.5 \cdot (2.25 - 1.1 - 1.2 \cdot 0.9346 + 2.25 - 1.1 - 1.2 \cdot 0.9524)}{1.06} = 0.0168;
\]
while on \( \{ \omega_3, \omega_4 \} \), \( c - c_1 B(2, 2) - c_2 B(2, 3) < 0 \) and

\[
S(1) = 0.
\]

## 9.3 Swaps, caps and floors

A few more interest rate derivatives are studied in this section.

### 9.3.1 Swaps and swaptions

A swap is an agreement between a payer and a receiver. The payer pays a fixed rate \( \kappa \) to, and meanwhile receives a floating rate \( r \) from the receiver. Their payments are based on a common principal. The payment is made each period during a time interval. A swap is said to be settled in arrears if one uses the spot rate \( r \) for the period just ended, or settled in advance if one uses the spot rate \( r \) for the period about to begin. With an ordinary swap the initial floating rate payment is based on the spot rate at the time when the agreement is made. With a forward start swap the initial floating rate payment is based on the spot rate subsequent to the one when the agreement is made. We will mainly focus on payer forward start swap on principal one settled each period in arrears. Other cases can be handled similarly. The value of a swap is the expected present value of the net cash flow, i.e.

\[
S(t) = E_Q \left\{ \sum_{s=\tau}^{\tau'} [r(s) - \kappa] \frac{B(t)}{B(s)} \mid \mathcal{F}_t \right\}, \quad t < \tau \leq \tau' \leq T. \tag{9.11}
\]

This formula can be simplified as follows:

\[
S(t) = E_Q \left\{ \sum_{s=\tau}^{\tau'} \frac{1}{B(s - 1, s)} - 1 - \kappa \frac{B(t)}{B(s)} \mid \mathcal{F}_t \right\} = E_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s - 1, s)} \mid \mathcal{F}_t \right] - (1 + \kappa) E_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s)} \mid \mathcal{F}_t \right] = E_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s - 1)} \mid \mathcal{F}_t \right] - (1 + \kappa) \sum_{s=\tau}^{\tau'} B(t, s) = \sum_{s=\tau}^{\tau'} B(t, s - 1) - (1 + \kappa) \sum_{s=\tau}^{\tau'} B(t, s) = B(t, \tau - 1) - \kappa \sum_{s=\tau}^{\tau'-1} B(t, s) - (1 + \kappa) B(t, \tau'). \tag{9.12}
\]
The \textit{forward swap rate} is the value of $\kappa$ in (9.11) and (9.12) which makes the time $t$ value of the forward swap in $[\tau, \tau']$ zero, denoted by

$$
\kappa(t, \tau, \tau') = \frac{B(t, \tau - 1) - B(t, \tau')}{B(t, \tau) + \cdots + B(t, \tau')}.
$$

(9.13)

A \textit{payer swaption} is a European call option on the time $\tau - 1$ value of the corresponding payer forward start swap, with exercise time $\tau - 1$ and exercise price zero. For $t < \tau$, the time $t$ value of the payer swaption is

$$
V_p(t) = E_Q \left\{ \frac{B(t)}{B(\tau - 1)} \left( E_Q \left[ \sum_{s=\tau}^{\tau'} B(s) \frac{B(\tau - 1)}{B(s)} (r(s) - \kappa) \bigm| \mathcal{F}_{t-1} \right] \right)^+ \bigg| \mathcal{F}_t \right\}.
$$

A \textit{receiver swaption} is defined similarly with its time $t$ value

$$
V_r(t) = E_Q \left\{ \frac{B(t)}{B(\tau - 1)} \left( E_Q \left[ \sum_{s=\tau}^{\tau'} B(s) \frac{B(\tau - 1)}{B(s)} (\kappa - r(s)) \bigm| \mathcal{F}_{t-1} \right] \right)^+ \bigg| \mathcal{F}_t \right\}.
$$

Note that

$$
V_p(t) - V_r(t) = E_Q \left\{ \frac{B(t)}{B(\tau - 1)} E_Q \left[ \sum_{s=\tau}^{\tau'} B(s) \frac{B(\tau - 1)}{B(s)} (r(s) - \kappa) \bigm| \mathcal{F}_{t-1} \right] \bigg| \mathcal{F}_t \right\}
$$

$$
= E_Q \left\{ \frac{B(t)}{B(\tau - 1)} \sum_{s=\tau}^{\tau'} B(s) \frac{B(\tau - 1)}{B(s)} (r(s) - \kappa) \bigg| \mathcal{F}_t \right\}
$$

$$
= E_Q \left\{ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s)} (r(s) - \kappa) \bigg| \mathcal{F}_t \right\},
$$

(9.14)

which is the time $t$ price of the forward start swap. Thus we have the following parity:

“payer swaption - receiver swaption = forward swap.”

By (9.12), another expression for the time $t$ price of the payer swaption is

$$
V_p(t) = E_Q \left\{ \frac{B(t)}{B(\tau - 1)} \left[ 1 - \kappa \sum_{s=\tau}^{\tau' - 1} B(\tau - 1, s) - (1 + \kappa) \right] B(\tau - 1, \tau') \bigg| \mathcal{F}_t \right\},
$$

which can be interpreted as this:

A payer swaption is same as a put option on a coupon bond, with exercise date $\tau - 1$ and exercise price one. The coupon has face value one and coupon rate $\kappa$. Similarly, a receiver swaption can be interpreted as a call option on a coupon bond.
To have another interpretation, suppose for each \( s = \tau, \tau + 1, \ldots, \tau' \), there is a call option with exercise time \( s \) and payoff \([\kappa(\tau - 1, \tau, \tau') - \kappa]^{+}\). The time \( \tau - 1 \) value of this portfolio consisting of a sequence of calls is given by

\[
S(t) = \mathbb{E}_Q \left\{ \frac{B(t)}{B(\tau - 1)} \mathbb{E}_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} (\kappa(\tau - 1, \tau, \tau') - \kappa)^{+} \mid \mathcal{F}_{\tau-1} \right] \mid \mathcal{F}_t \right\}
\]

\[
= \mathbb{E}_Q \left\{ \frac{B(t)}{B(\tau - 1)} (\kappa(\tau - 1, \tau, \tau') - \kappa)^{+} \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} \mid \mathcal{F}_{\tau-1} \right\}
\]

\[
= \mathbb{E}_Q \left\{ \frac{B(t)}{B(\tau - 1)} (\kappa(\tau - 1, \tau, \tau') - \kappa)^{+} \sum_{s=\tau}^{\tau'} B(\tau - 1, s) \mid \mathcal{F}_t \right\}
\]

\[
= \mathbb{E}_Q \left\{ \frac{B(t)}{B(\tau - 1)} \left[ 1 - B(\tau - 1, \tau') - \kappa \sum_{s=\tau}^{\tau'} B(\tau - 1, s) \right]^{+} \mid \mathcal{F}_t \right\}, \quad (9.15)
\]

which is just the time \( t \) price of the payer swaption.

### 9.3.2 Caps and floors

A caplet is a European call option on the spot rate \( r(\tau) \) at a specific time \( \tau \) and with a specified exercise price \( \kappa \). Thus its time \( t \) price is

\[
S(t) = \mathbb{E}_Q \{ [r(\tau) - \kappa]^{+} B(t)/B(\tau) \mid \mathcal{F}_t \} = B(t, \tau) \mathbb{E}^{\tau} \{ [r(\tau) - \kappa]^{+} \mid \mathcal{F}_t \}.
\]

A cap is a sequence of caplets which have a common exercise price. Like swaps, there are ordinary and forward start caps, depending on whether the initial caplet corresponds to the current spot rate. The time \( t \) price of a forward start cap in \([\tau, \tau']\) settled in arrears is given by

\[
S(t) = \sum_{s=\tau}^{\tau'} \mathbb{E}_Q \{ [r(s) - \kappa]^{+} B(t)/B(s) \mid \mathcal{F}_t \}, \quad t < \tau \leq \tau' \leq T. \quad (9.16)
\]

A floorlet is defined in the same way as a caplet, but as a put option on the spot rate. A floor is a sequence of floorlets. The time \( t \) price of a forward start floor in \([\tau, \tau']\) settled in arrears is given by

\[
S(t) = \sum_{s=\tau}^{\tau'} \mathbb{E}_Q \{ [\kappa - r(s)]^{+} B(t)/B(s) \mid \mathcal{F}_t \}, \quad t < \tau \leq \tau' \leq T. \quad (9.17)
\]

It is obvious that the price of a cap minus the price of a floor is equal to the price of a swap.

Furthermore, a caption is a call or put option on a forward cap; while a floortion is a call or put option on a forward floor. These derivatives are examples of compound options.