**STOR 890, Spring 2011, Homework 3**

**Note:** Try these problems yourselves, and we will organize a meeting before the final exam to discuss them.

**Assume the BS market in problems (1) — (4).**

(1) Suppose a stock price is currently $50, the stock return has mean 18% and volatility 30%. What is the probability distribution for the stock price in two years? Specify the mean and standard deviation of the distribution.

(2) With the assumptions in (1), specify the strike price $K$ for a European call that becomes mature in two years, such that with probability 0.7 (under the physical measure $P$) the option is exercised.

(3) A financial institution plans to offer a security, written on a stock $S$, with pay-off $S_T^2$ at time $T$. Find the expression of time $t$ price of the security.

(4) Suppose a call option on a stock has a market price $2.50, and the stock price is $15, the exercise price is $13, the time to maturity is 3 months, and the annual interest rate is 5%. Find the implied volatility. **Hint:** Use any software convenient to you to conduct the numerical search.

(5) Find the time $t$ price of a European option that pays $D$ dollars if $\min\{S_{1T}, S_{2T}\} > K$ and zero otherwise at the maturity $T > t$, where we assume the continuous time model (under the physical measure $P$):

$$dS_{it} = S_{it} \left( \mu_i \; dt + \sigma_i \; dW_{it} \right)$$

for $i = 1, 2$, with independent Brownian motions $W_1$ and $W_2$.

(6) Let $W^{(i)} \triangleq \{W^{(i)}_t\}_{t \in [0,T]}, \; i = 1, \ldots, d$ be $d$ independent 1D standard Brownian motions. Recall that for each $i$, the quadratic variation satisfies $\langle W^{(i)}, W^{(i)} \rangle_t = t$, or we can write this as $dW^{(i)}_t \; dW^{(i)}_t = dt$. Show in what follows that for $i \neq j$, the cross variation process satisfies $dW^{(i)}_t \; dW^{(j)}_t = 0$. More specifically, for a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$, let $\Delta = \max\{ |t_k - t_{k-1}| : \; k = 1, \ldots, n \}$, and define $\xi_{ij}(\Delta)$ as:

$$\xi_{ij}(\Delta) = \sum_{k=1}^{n} \left[ W^{(i)}_{t_k} - W^{(i)}_{t_{k-1}} \right] \left[ W^{(j)}_{t_k} - W^{(j)}_{t_{k-1}} \right].$$

(6a) Show that $E[\xi_{ij}(\Delta)] = 0$.

(6b) Show that $\text{Var}[\xi_{ij}(\Delta)] = E[\xi_{ij}(\Delta)]^2 \leq T \; \Delta$.

(6c) Show that (6a) and (6b) imply $\xi_{ij}(\Delta)$ converges to 0 in probability as $\Delta \to 0$. 


(7) Let $W$ be a standard Brownian motion and define two processes $X$, $Y$ by

$$dX_t = dt + 2\, dW_t$$

and

$$Y_t = e^{X_t} \int_0^t e^{-X_u} dW_u.$$  

(7a) Calculate $EY_t$.  
(7b) Calculate $E(Y_t^2)$.  

**Note:** Apply Itô’s formula, perhaps its multidimensional version if needed.  

(8) Consider a special case of the HJM model under a risk neutral measure $Q$ (cf. Section 19.2.3) with a constant volatility $\sigma(t, \tau) \equiv \sigma > 0$, $\forall \ 0 \leq t \leq \tau \leq T$. We want to price a European call with maturity $\tau < T$ and strike price $K$, written on a zero-coupon bond $B(\cdot, T)$.  

(8a) Show that the forward rate satisfies

$$f(t, u) = f(0, u) + \sigma^2 (tu - t^2)/2 + \sigma W_t.$$  

(8b) Show that the zero-coupon bond satisfies

$$B(\tau, T) = \frac{B(0, T)}{B(0, \tau)} \exp[-\sigma(T - \tau) W_\tau - \sigma^2 \tau T(T - \tau)/2].$$

(8c) Find the expression of the short rate $r_t$.  
(8d) Show that the time 0 price of the call option is given by

$$C_0 = B(0, T) \Phi(d) - K B(0, \tau) \Phi(d - \sigma \sqrt{T(\tau - T)}),$$

where $\Phi(\cdot)$ is the cdf of $N(0,1)$ and

$$d = \frac{\sigma \sqrt{T(\tau - T)}}{2 \sigma \sqrt{T(\tau - T)}} - \frac{\log[K B(0, \tau)/B(0, T)]}{\sigma \sqrt{T(\tau - T)}}.$$  

(9) For $t \in [0, T]$, a market contains a riskless bank account $B$ that satisfies $B_t = e^{rt}$ with a constant rate $r > 0$, two risky assets $S^{(i)}$, $i = 1, 2$ that satisfy the $P$-dynamics

$$dS_t^{(i)} / S_t^{(i)} = \mu_i \, dt + \sigma_i \, dW_t^{(i)},$$

with constants $\mu_i \in \mathbb{R}$, $\sigma_i > 0$ and two independent Brownian motions $W^{(i)}$.  

(9a) Define

$$Z_t = \exp \left[-\theta_1 W_t^{(1)} - \theta_2 W_t^{(2)} - (\theta_1^2 + \theta_2^2) \, t/2 \right],$$
with \( \theta_i = (\mu_i - r)/\sigma_i, \ i = 1, 2 \). Show that \( \{Z_t\} \) is a \( P \)-martingale with respect to the filtration \( \{\mathcal{F}_t\}_{t \in [0,T]} \) generated by \((W^{(1)}, W^{(2)})\).

(9b) Define a risk neutral measure \( Q \) by \( \frac{dQ}{dP} = Z_T \). Then \( \{e^{-rt}S_t^{(i)}\}_{t \in [0,T]}, \ i = 1, 2 \) are \( Q \)-martingales, and an option \( g \left( S_T^{(1)}, S_T^{(2)} \right) \) with maturity \( T \) can be priced at time \( t \) as

\[
V_t = P \left( t; S_t^{(1)}, S_t^{(2)} \right) = E_Q \left[ e^{-r(T-t)}g \left( S_T^{(1)}, S_T^{(2)} \right) \mid \mathcal{F}_t \right].
\]

You do not have to justify it, but will need to find more explicit expressions of this pricing formula in (9c).

(9c) Consider the option \( g \left( S_T^{(1)}, S_T^{(2)} \right) = \left( S_T^{(1)} - S_T^{(2)} \right)^+ \), i.e. giving the holder the right (not obligation) to exchange one unit of \( S_T^{(2)} \) for one unit of \( S_T^{(1)} \) at time \( T \). Derive a formula for \( V_t \) similar to the BS option pricing formula, in terms of \( S_t^{(1)}, S_t^{(2)}, r, \sigma_1, \sigma_2 \) and the cdf \( \Phi(\cdot) \).