(1) What are the differences between taking a long position in a forward contract with the forward price $20 and buying a call option with a strike price $20?

**Answer:** (i) No cost for entering a forward contract; but need to pay a price for buying a call option. (ii) At the maturity $T$, the payoff for the forward contract is $S(T) - 20$, but the payoff for the call option is $[S(T) - 20]^+$. 

(2) Assume an annual interest rate 5% and quarterly compounding. Which of the following two plans is a better investment? Why?

- Plan A: investing $10,000 today, receiving $5,000, $4,500, $4,500 after 3, 6, 9 months respectively;
- Plan B: investing $11,480 today, receiving $5,500, $5,000, $5,000 after 3, 6, 9 months respectively.

**Answer:** Let $r = 0.05/4$ be the quarterly compounding rate and $b = \frac{1}{1 + r}$. Motivated by the time value of money, consider the present values of cash flows driven by Plan A and Plan B respectively, with the difference 

$$PV_A - PV_B = 1480 - 500 \left(b + b^2 + b^3 \right) \approx 16.7 > 0.$$ 

Therefore, Plan A is a better investment.

(3) Consider a single period market in which a bank account has an interest rate $r = 0.05$.

(3a) If a stock $S$ follows a binomial tree with “up” factor $u = 1.05$ and “down” factor $d = 0.96$, can we determine the time-0 value $V(0)$ of a contingent claim $g(S(1))$? Explain why.

**Answer:** If an EMM $Q$ existed, it would be given by 

$$Q(S(1)/S(0) = u) \triangleq q = \frac{1 + r - d}{u - d} = 1$$

in this case due to $1 + r = u$. Such a degenerate measure $Q$ cannot be equivalent to the physical measure $P$. Hence there exists an arbitrage opportunity. Here is an example: Short $S$ and deposit the money in $B$ at time $t = 0$, then buy back $S$ and return it to the lender at time $t = 1$. It’s obvious this strategy will never lose, and have a positive profit $(1 + r - d) S(0)$ if $S(1)/S(0) = d$.

(3b) If a stock $S$ follows a trinomial tree with “up” factor $u = 1.08$, a “flat” factor $f = 1$, and “down” factor $d = 0.95$, can we determine the time-0 value $V(0)$ of a contingent claim $g(S(1))$? Explain why.
Answer: No arbitrage in this situation but the market is incomplete. I suggest that you review various concepts and results in Lecture 3: no arbitrage, Law of One Price, marketability of a contingent claim, market completeness, existence and uniqueness of EMM Q, etc. In particular, in the current case of incomplete market the contingent claim \( g(S(1)) \) need not be marketable hence we need not be able to determine the value \( V(0) \) via the risk neutral valuation principle.

(4) Suppose a stock \( S \) has the initial price \( S(0) = 5 \) at \( t = 0 \), and a bank account \( B \) has the constant interest rate \( r = 0.02 \) in each time period. A dividend \( D \) associated with \( S \) is issued at \( t = 1 \). With the same maturity \( T = 2 \) and strike price \( c = 5.1 \), a call and a put turn out to have an identical price at \( t = 0 \). Determine the dividend value \( D \).

Answer: Since \( [S(2) - c]_+ - [c - S(2)]_+ = S(2) - c \), we have

\[
0 = C(0) - P(0) = \frac{E_Q S(2) - c}{(1+r)^2} = S(0) - \frac{D(1)}{1+r} - \frac{c}{(1+r)^2}
\]

where \( C(0) \) and \( P(0) \) denote the prices of call and put respectively at \( t = 0 \). Hence

\[
D(1) = (1+r)S(0) - \frac{c}{1+r} = 0.1.
\]

(5) Consider a single-period model with a bank account \( B \) and a stock \( S \). Assume \( S(0) = $100 \) and three possible scenarios: $150, $100, $80. Defined on \( S \) and with maturity \( T = 1 \), the price of a put with strike price $102 is $8, and the price of a put with strike price $95 is $5.

(5a) Show that the interest rate \( r = 1/14 \).

Answer: Let \( q_i, i = 1, 2, 3 \) denote the risk neutral probabilities. Set up 4 equations with variables \( q_1, q_2, q_3 \) and \( r \):

\[
\begin{align*}
q_1 + q_2 + q_3 &= 1 \\
150q_1 + 100q_2 + 80q_3 &= 100(1+r) \\
(102 - 100)q_2 + (102 - 80)q_3 &= 8(1+r) \\
(95 - 80)q_3 &= 5(1+r)
\end{align*}
\]

which will give rise to \( r = 1/14 \).

(5b) What portfolio would end up with the payoff of $15 at \( t = 1 \) regardless of possible scenarios of \( S \)?

Answer: The portfolio with no shares of \( S \) and $14 in \( B \) will do ...

(5c) Consider a call option on \( S \) with maturity \( T = 1 \) and strike price \( 80 < c < 100 \). Construct a portfolio \( (h_0, h_1) \) with an initial balance \( h_0 \) in \( B \) and \( h_1 \) shares of \( S \) such that it replicates the call option.
Answer: With $1 + r = 15/14$, set 3 equations with variables $h_0$, $h_1$ and $c$:

\[
\begin{align*}
150 - c &= (1 + r)h_0 + 150h_1 \\
100 - c &= (1 + r)h_0 + 100h_1 \\
0 &= (1 + r)h_0 + 80h_1
\end{align*}
\]

which will give rise to $h_0 = -14 \cdot 16/3$, $h_1 = 1$ and $c = 80$. However, $c = 80$ violates the assumption $80 < c < 100$. Hence no such portfolio exists.

(5d) Determine the strike price $c$ in (5c).

Answer: See (5c).

(5e) Without the assumption $80 < c < 100$ given in (5c), can you still obtain the same results in (5c) and (5d). Explain why.

Answer: If $c \leq 80$, then the 3 equations for the replicating portfolio becomes

\[
\begin{align*}
150 - c &= (1 + r)h_0 + 150h_1 \\
100 - c &= (1 + r)h_0 + 100h_1 \\
80 - c &= (1 + r)h_0 + 80h_1
\end{align*}
\]

which implies $h_1 = 1$ and

\[(1 + r)h_0 + c = 0\]

that will lead to infinitely many solutions. To sum up, either no solutions exist if $c \in (80, 100)$ or solutions are not unique if $c \leq 80$. Question: What if $100 < c < 150$?

(6) Consider a binomial tree model with $T = 2$. Identify $\omega_1 = uu$, $\omega_2 = ud$, $\omega_3 = du$ and $\omega_4 = dd$. Let $S(0) = 1$, $u = 1.06$, $d = 0.95$; $B(0) = B(1) = 1$ and

\[
B(2) = \begin{cases} 
1.03, & \text{on } \{\omega_1, \omega_2\} \\
1.04, & \text{on } \{\omega_3, \omega_4\}
\end{cases}
\]

(6a) Under the risk-neutral probability measure $Q$, let $q_0$ be the probability of “$S$ goes up” in the first period, $q_1$ be the probability of “$S$ goes up” in the second period on $\{\omega_1, \omega_2\}$, and $q_2$ be the probability of “$S$ goes up” in the second period on $\{\omega_3, \omega_4\}$. Find $q_0$, $q_1$ and $q_2$.

Answer: Set

\[
\begin{align*}
1.06q_1 + 0.95(1 - q_1) &= 1.03 \\
1.06q_2 + 0.95(1 - q_2) &= 1.04
\end{align*}
\]
we have \( q_1 = \frac{8}{11} \) and \( q_2 = \frac{9}{11} \). Furthermore, \( q_0 = \frac{1 - 0.95}{1.06 - 0.95} = \frac{5}{11} \).

(6b) Consider a forward contract and a futures contract on the stock \( S \), both start at \( t = 0 \) and end at \( T = 2 \). Do they have the same value at \( t = 0 \)? Do they have the same price? Why?

**Answer:** First, they have the same value zero at \( t = 0 \). Next, they need not have the same price. More specifically, the forward price is given by

\[
FO(0) = S(0) \left[ EQ \frac{1}{B(2)} \right]^{-1} = \left( \frac{q_0}{1.03} + \frac{1 - q_0}{1.04} \right)^{-1} = ...
\]

To obtain the futures price, we have

\[
FU(2, 2) = S(2);
\]

\[
S(1)/B(1) = FU(1, 2) EQ \left[ \frac{1}{B(2)} \mid F_1 \right]
\]

which gives rise to

\[
FU(1, 2) = \begin{cases} 1.06 \cdot 1.03, & \text{on } \{\omega_1, \omega_2\} \\ 0.95 \cdot 1.04, & \text{on } \{\omega_3, \omega_4\} \end{cases}
\]

Moreover,

\[
FU(0, 2) = EQ[FU(1, 2)] = ...
\]

(6c) For a European call option and a European put option on the futures \( FU(1, 2) \), specify the strike price \( c \) such that these two European options have the same value at \( t = 0 \).

**Answer:** Set

\[
EQ[FU(1, 2) - c]^+ = EQ[c - FU(1, 2)]^+
\]

which implies

\[
c = EQ[FU(1, 2)] = FU(0, 2)
\]

with the same value as we obtained in (6b).