Option Pricing

STOR 890 (2011)

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Option pricing: a fundamental problem in studying derivatives

Why need it? The fair price is a basis for trading, risk management (hedging), etc.

Methods: (i) analytical (solving differential equations); (ii) probabilistic (computing expected values in a “risk neutral” world).

Toy example: a single period binomial tree for illustration
Basic setting

- **Time horizon**: a single period with $t = 0$ (now) and $T = 1$ (future)
- **Risky asset**: stock $S = \{S_0, S_1\}$ with $S_0 = \$2$; it will go up by a factor $u = 1.07$ (up) with probability $p = 0.6$, or go down by a factor $d = 0.92$ (down) with probability $1 - p = 0.4$.
- **Risk-free asset**: bank account $B = \{B_0, B_1\}$ with $B_0 = \$1$ and a fixed interest rate $r = 0.06$. 
Call option on $S$

- **Call option**: price $C = \{C_0, C_1\}$
- **Contract**: signed at $t = 0$, to expire at $T = 1$, with strike price $K = $2.05
- **Pay-off**: At $t = 0$, you can long the contract $C$ by paying the counter party $C_0$; at $T = 1$, you would exercise the option and receive the pay-off $S_1 - K$ if $S_1 > K$, or do nothing (thus receive zero pay-off) if $S_1 \leq K$.
- **Option pricing**: What is the fair price $C_0$?
- **Strategy**: Construct a replicating portfolio $P = \{P_0, P_1\}$ that consists of $S$ and $B$ such that $P_0 = C_0$ and $P_1 = C_1$, i.e. $P$ and $C$ have exactly the same values at any time with any possible scenarios.
Step 1

Stock tree:

\[ S_1 = \begin{cases} 
S_0 \cdot u = 2 \times (1.07) = 2.14, & \text{with probability 0.6} \\
S_0 \cdot d = 2 \times (0.92) = 1.84, & \text{with probability 0.4}
\end{cases} \]
Note: \( C_1 = P_1 = (1 + r) x + S_1 y \), where \( x \) and \( y \) are quantities of \( B \) and \( S \) respectively held in the portfolio \( P \) from \( t = 0 \) to \( T = 1 \).

For the two scenarios “up” and “down”, the above equation will turn to

\[
2.14 - 2.05 = 1.06 x + 2.14 y, \\
0 = 1.06 x + 1.84 y,
\]

which leads the solution \( (x, y) \approx (-0.52, 0.3) \). Therefore,

\[
C_0 = P_0 = x + 2 y = -0.52 + 0.6 = 0.08.
\]

Conclusion: Hypothetically, if we construct a portfolio \( P \) by borrowing $0.52 from the bank and buy 0.3 shares of \( S \), then this portfolio would yield the same value as the call option \( C \) at both \( t = 0 \) and \( T = 1 \). Hence the fair price for the call option is \( C_0 = $0.08 \).
The replicating portfolio tree:

- $P_0 = 0.08$
- $P_1 = 0.09$
- $P_1 = 0$
A probabilistic method for option pricing: define a risk-neutral probability \( q \) under which the option price is computed as the expected value of discounted pay-off, denoted by \( C_0 = E_q [(1 + r)^{-1}C_1] \). In this example, we have a Recipe: let

\[
q = \frac{1 + r - d}{u - d} = \frac{1.06 - 0.92}{1.07 - 0.92} = \frac{14}{15};
\]

then

\[
C_0 = E_q [(1 + r)^{-1}C_1] = \frac{q \cdot 0.09 + (1 - q) \cdot 0}{1 + r} = \frac{14}{15} \frac{0.09}{1.06} \approx 0.08.
\]
This method applies to binomial trees with multiple periods via backward induction (see Lecture 2).

Exercise: Define a put option under the same conditions as those given in this example, and find its fair price.

Basic idea of replicating portfolios: There are often many different ways to obtain the same outcome ... That is why so many derivatives are invented.