Lecture Notes on Discrete-time Finance

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Most parts in the lecture notes were based on the materials in Pliska’s excellent book *Introduction to Mathematical Finance* (1997, Blackwell Publishers Inc.) except for the Black-Scholes option pricing formula, and the implied volatility trees.

The required mathematical background is minimal: calculus, linear algebra, calculus-based probability and statistics. Some knowledge of elementary optimization and financial engineering would certainly be useful, but not crucial. In particular, no knowledge of stochastic calculus is needed. Instead, the focus will be on discrete-time models, such as binomial trees. We plan to take the approach to as far as we can go, including the treatment for stock and fixed-income derivatives. This approach might lose the beauty or even the simplicity of certain formulas derived via stochastic calculus. Nevertheless, it enables us to

(a) present almost all useful results by using only elementary mathematics;
(b) build up our intuition more quickly;
(c) learn some numerical computation methods directly.

We will attempt at introducing the related continuous-time limit in the late stage of each chapter, yet we only do so via some basic calculation without invoking weak convergence of stochastic processes, i.e. we are limited to verify the convergence in finite-dimensional distributions, but not the “tightness”.
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Chapter 1

Model Specifications

1.1 Asset price dynamics

To model the financial market statistically, several basic elements are needed.

- A finite sample space \( \Omega = \{ \omega_1, \ldots, \omega_K \} \).
- A probability measure \( P \) on \( \Omega \) with \( P(\omega) > 0 \ \forall \ \omega \in \Omega \).
- A filtration \( \mathcal{F} = \{ \mathcal{F}_t, t = 0, 1, \ldots, T \} \) with \( \mathcal{F}_{t-1} \subseteq \mathcal{F}_t, t = 1, \ldots, T \), where \( \mathcal{F}_t \) contains the information about the financial market available to the investors at time \( t \). Usually, \( t = 0, 1, \ldots, T \) represent \( T + 1 \) trading dates. Since \( T < \infty \), this is called a finite horizon model or a multiperiod model.
- A riskless bank account process \( B = \{ B(t), t = 0, 1, \ldots, T \} \), where \( B(0) = 1 \) and \( B(t) > 0 \ \forall t \). \( B(t) \) is thought of as the time \( t \) value of a saving account when $1 is deposited at time 0. Hence \( B(t) \) is nondecreasing in \( t \). Moreover, the quantity \( r(t) = \frac{[B(t) - B(t-1)]}{B(t-1)} \) is thought of as the interest rate pertaining to the time interval \( (t-1, t] \).
- \( N \) risky security processes \( S_n = \{ S_n(t), t = 0, 1, \ldots, T \}, n = 1, \ldots, N \), where \( S_n(t) \geq 0 \) is thought of as the time \( t \) price of risky security \( n \) (e.g. stock or bond).

Note that \( B, S_1, \ldots, S_N \) are considered to be stochastic processes, i.e. for each \( t \), \( B(t), S_1(t), \ldots, S_N(t) \) are all functions of \( \omega \). To ease the notation, the dependence on \( \omega \) is usually not shown unless necessary. Furthermore, \( B, S_1, \ldots, S_N \) are assumed to be adapted to the filtration \( \mathcal{F} \). A stochastic process \( \{ X(t) \} \) is said to be adapted to the filtration \( \mathcal{F} \) if for each \( t \), the random variable \( X(t) \) is measurable with respect to \( \mathcal{F}_t \), i.e. the information about \( X(t) \) is contained in \( \mathcal{F}_t \).
1.2 Trading strategies

A trading strategy $h = (h_0, h_1, \ldots, h_N)$ is a vector of processes $h_n = \{h_n(t), \ t = 1, \ldots, T\}$, $n = 0, 1, \ldots, N$. Note that $h_n(0)$ is not specified, because for $n = 1, \ldots, N$, $h_n(t)$ is interpreted as the number of units (e.g. shares of stock) that the investor owns (i.e. carries forward) from time $t - 1$ to time $t$, whereas $h_0(t)$ $B(t - 1)$ represents the amount of money invested in the bank account at time $t - 1$. A negative value of $h_n(t)$ corresponds to borrowing money from the bank (when $n = 0$) or selling short security $n$ (when $n = 1, \ldots, N$). $h$ is also called a portfolio.

A trading strategy is a rule that specifies the investor’s position in each security $n$ at each time $t$ and in each state of the world $\omega$. In general, this rule should allow the investor to choose a position in the securities based on the available information thus far without “looking into the future”. This is done by introducing the concept of predictability.

A stochastic process $\{X(t)\}$ is said to be predictable with respect to the filtration $\mathcal{F}$ if for each $t = 1, 2, \ldots$ the random variable $X(t)$ is measurable with respect to $\mathcal{F}_{t-1}$. (Note: “predictable” implies “adapted”, why?) In what follows we assume that each component of a trading strategy $h$ is a predictable process.

1.3 Value processes, gain processes and self-financing strategies

The value process $V = \{V(t), \ t = 0, 1, \ldots, T\}$ consists of the initial value of the portfolio

$$V(0) = h_0(1)B(0) + \sum_{n=1}^{N} h_n(1)S_n(0)$$  \hspace{1cm} (1.1)

and the time $t \ (t \geq 1)$ value of the portfolio

$$V(t) = h_0(t)B(t) + \sum_{n=1}^{N} h_n(t)S_n(t)$$  \hspace{1cm} (1.2)

before any transactions are made at the same time. (Note: $V$ is adapted, why?)

Denote $\Delta S_n(t) = S_n(t) - S_n(t - 1)$ for the increment of $S_n$ between $t - 1$ and $t$. Then $h_n(t) \Delta S_n(t)$ represents the one-period gain or loss due to the ownership of $h_n(t)$ units of security $n$ between $t - 1$ and $t$; and $\sum_{u=1}^{t} h_n(u) \Delta S_n(u)$ represents the cumulative gain or loss up to time $t$ due to the investment of security $n$. Hence

$$G(t) = \sum_{u=1}^{t} h_0(u) \Delta B(u) + \sum_{n=1}^{N} \sum_{u=1}^{t} h_n(u) \Delta S_n(u)$$  \hspace{1cm} (1.3)
represents the cumulative gain or loss of the portfolio up to time $t$. $G = \{G(t), \ t = 1, \ldots, T\}$ is called a gain process (also adapted, why?).

A trading strategy is said to be self-financing if for $t = 1, \ldots, T-1$,

$$V(t) = h_0(t+1) \ B(t) + \sum_{n=1}^N h_n(t+1) \ S_n(t).$$

(1.4)

The motivation is that the LHS represents the time $t$ value of the portfolio just before any transactions (i.e. any changes of ownership positions) take place at that time, while the RHS represents the time $t$ value of the portfolio right after any transactions (i.e. before the portfolio is carried forward to $t+1$). In general, the two values can be different, which means at time $t$ some money is added to or withdrawn from the portfolio. However, for many applications this cannot happen at other than $t = 0$ and $t = T$, and so it leads to the above definition. For a self-financing strategy, any change in the portfolio’s value is due to a gain or loss in the investments.

It is straightforward to check (do it yourself) the following: A strategy $h$ is self-financing if and only if

$$V(t) = V(0) + G(t), \ t = 1, \ldots, T.$$  

(1.5)

Note that $V(1) = V(0) + G(1)$ always holds (why?).

### 1.4 Discounted prices

For the studies of finance modelling, what really matters is the behavior of the security prices relative to each other, rather than their absolute behavior. Hence we are interested in normalized versions of the security prices with respect to the price of a standard security — usually using the bank account for convenience. In general, some other riskless securities could be chosen as the “yardstick”, called the numeraire.

Define the discounted price processes $S^*_n = \{S^*_n(t), \ t = 0, 1, \ldots, T\}, \ n = 1, \ldots, N$ by

$$S^*_n(t) = S_n(t)/B(t), \ t = 0, 1, \ldots, T;$$

(1.6)

the discounted value process $V^* = \{V^*(t), \ t = 0, 1, \ldots, T\}$ by

$$V^*(0) = h_0(1) + \sum_{n=1}^N h_n(1)S^*_n(0)$$

(1.7)

and

$$V^*(t) = h_0(t) + \sum_{n=1}^N h_n(t)S^*_n(t);$$

(1.8)
and the discounted gain process $G^* = \{G^*(t), \ t = 1, \ldots, T\}$ by

$$G^*(t) = \sum_{n=1}^{N} \sum_{u=1}^{t} h_n(u) \Delta S_n^*(u), \quad t = 1, \ldots, T. \tag{1.9}$$

Note in particular, $B^*(t) = 1$ and $\Delta B^*(t) = 0$, $\forall t$. It is also easy to check:

$$V^*(t) = V(t)/B(t), \quad t = 0, 1, \ldots, T; \tag{1.10}$$

and that a strategy $h$ is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t), \quad t = 1, \ldots, T. \tag{1.11}$$
Chapter 2

Binomial Trees: an Example

2.1 Illustration of concepts introduced in Lecture 1

• Figure 2.1, called a binomial tree, illustrates how a stock $S = \{S(t), t = 0, 1, 2, 3\}$ changes. We suppress the subscript $n$ since $N = 1$. Assume $S(0) = \$2$, in each period the stock price either goes up by the factor $u = 1.07$ with probability $p = 0.6$, or goes down by the factor $d = 0.92$ with probability $1 - p = 0.4$, i.e. the moves over time are iid Bernoulli random variables. Hence $S(t) = S(0)u^{n_t}d^{3-n_t}$, $t = 0, 1, 2, 3$, where $n_t$ represents the number of up moves up to $t$.

The sample space $\Omega = \{\omega_1, \ldots, \omega_8\}$, where each $\omega_k$ corresponds to a path, e.g. $\omega_6$ can be identified as the path “down-up-down” (dud), etc. Each probability $P(\omega_k)$ can be calculated easily, e.g. $P(\omega_6) = p(1 - p)^2 = 0.096$.

• Suppose the bank account process $B$ is deterministic with a constant interest rate $r(t) \equiv 0.06$. In general, many different filtrations $\mathcal{F}$ can be defined in which each $\mathcal{F}_t$ contains the history of the stock up to $t$ and perhaps some other information. This will become more useful later in this course. For now, we simply adopt the following particular filtration generated by the stock process $S$: Each $\mathcal{F}_t$ involves precisely the history of $S$ up to $t$ and no additional information. More specifically, each $\mathcal{F}_t$ is equivalent to a partition $\mathcal{P}_t$ of $\Omega$ consisting of subsets of $\Omega$ with “no omission and no overlap”. The partitions can be specified by paths:

\[
\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\} \\
= \{\{uuu, uud, udu,udd\}, \{duu, dud, ddu, ddd\}\}; \\
\]

\[
\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}\} \\
= \{\{uuu, uud\}, \{udu, udd\}, \{duu, dud\}, \{ddu, ddd\}\}; \\
\]
Figure 2.1: Stock price tree
and

\[ \mathcal{P}_3 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\} \]
\[ = \{uuu, uud, udu, ud, d, ddu, dud, ddd\}. \]

As \( t \) increases, the partition \( \mathcal{P}_t \) becomes finer and \( \mathcal{F}_t \) reveals more information about the evolution of stock \( S \).

- The value process, gain process and their discounted versions depend on a given trading strategy (portfolio process). For each \( t \), \( B(t) = (1 + 0.06)^t \) and the portfolio is \((h_0(t), h_1(t))\).

Following (1.1) and (1.2), we have the value process

\[ V(0) = h_0(1) + 2.00 \cdot h_1(1), \]
\[ V(1) = \begin{cases} 
(1 + 0.06) \cdot h_0(1) + 2.14 \cdot h_1(1), & \text{on } \{\omega_1, \omega_2, \omega_3, \omega_4\} \\
(1 + 0.06) \cdot h_0(1) + 1.84 \cdot h_1(1), & \text{on } \{\omega_5, \omega_6, \omega_7, \omega_8\} 
\end{cases} \]
\[ V(2) = \begin{cases} 
(1 + 0.06)^2 \cdot h_0(2) + 2.29 \cdot h_1(2), & \text{on } \{\omega_1, \omega_2\} \\
(1 + 0.06)^2 \cdot h_0(2) + 1.97 \cdot h_1(2), & \text{on } \{\omega_3, \omega_4\} \text{ or } \{\omega_5, \omega_6\} \\
(1 + 0.06)^2 \cdot h_0(2) + 1.69 \cdot h_1(2), & \text{on } \{\omega_7, \omega_8\} 
\end{cases} \]

and

\[ V(3) = \begin{cases} 
(1 + 0.06)^3 \cdot h_0(3) + 2.45 \cdot h_1(3), & \text{on } \{\omega_1\} \\
(1 + 0.06)^3 \cdot h_0(3) + 2.11 \cdot h_1(3), & \text{on } \{\omega_2\} \text{ or } \{\omega_3\} \text{ or } \{\omega_5\} \\
(1 + 0.06)^3 \cdot h_0(3) + 1.81 \cdot h_1(3), & \text{on } \{\omega_4\} \text{ or } \{\omega_6\} \text{ or } \{\omega_7\} \\
(1 + 0.06)^3 \cdot h_0(3) + 1.56 \cdot h_1(3), & \text{on } \{\omega_8\}. 
\end{cases} \]

The gain process in (1.3) can be written (in this example) as

\[ G(t) = G(t - 1) + h_0(t) \cdot \Delta B(t) + h_1(t) \cdot \Delta S(t). \]

Hence we have

\[ G(1) = \begin{cases} 
0.06 \cdot h_0(1) + 0.14 \cdot h_1(1), & \text{on } \{\omega_1, \omega_2, \omega_3, \omega_4\} \\
0.06 \cdot h_0(1) - 0.16 \cdot h_1(1), & \text{on } \{\omega_5, \omega_6, \omega_7, \omega_8\} 
\end{cases} \]
\[ G(2) = \begin{cases} 
0.06 \cdot h_0(1) + 0.14 \cdot h_1(1) + 0.06 \cdot h_0(2) + 0.15 \cdot h_1(2), & \text{on } \{\omega_1, \omega_2\} \\
0.06 \cdot h_0(1) + 0.14 \cdot h_1(1) + 0.06 \cdot h_0(2) - 0.17 \cdot h_1(2), & \text{on } \{\omega_3, \omega_4\} \\
0.06 \cdot h_0(1) - 0.16 \cdot h_1(1) + 0.06 \cdot h_0(2) + 0.13 \cdot h_1(2), & \text{on } \{\omega_5, \omega_6\} \\
0.06 \cdot h_0(1) - 0.16 \cdot h_1(1) + 0.06 \cdot h_0(2) - 0.15 \cdot h_1(2), & \text{on } \{\omega_7, \omega_8\} 
\end{cases} \]
and

\[
G(3) = \begin{cases} 
0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) + 0.15 h_1(2) \\
+ 0.07 h_0(3) + 0.16 h_1(3), & \text{on } \{\omega_1\} \\
0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) + 0.15 h_1(2) \\
+ 0.07 h_0(3) - 0.18 h_1(3), & \text{on } \{\omega_2\} \\
0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) - 0.17 h_1(2) \\
+ 0.07 h_0(3) + 0.14 h_1(3), & \text{on } \{\omega_3\} \\
0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) - 0.17 h_1(2) \\
+ 0.07 h_0(3) - 0.16 h_1(3), & \text{on } \{\omega_4\} \\
0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) + 0.13 h_1(2) \\
+ 0.07 h_0(3) + 0.14 h_1(3), & \text{on } \{\omega_5\} \\
0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) + 0.13 h_1(2) \\
+ 0.07 h_0(3) - 0.16 h_1(3), & \text{on } \{\omega_6\} \\
0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) - 0.15 h_1(2) \\
+ 0.07 h_0(3) + 0.12 h_1(3), & \text{on } \{\omega_7\} \\
0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) - 0.15 h_1(2) \\
+ 0.07 h_0(3) - 0.13 h_1(3), & \text{on } \{\omega_8\}. 
\end{cases}
\]

We now look at the condition (1.4) for self-financing portfolios. For \( t = 1 \),

\[
\begin{align*}
1.06 h_0(1) + 2.14 h_1(1) &= 1.06 h_0(2) + 2.14 h_1(2), & \text{on } \{\omega_1, \omega_2, \omega_3, \omega_4\} \\
1.06 h_0(1) + 1.84 h_1(1) &= 1.06 h_0(2) + 1.84 h_1(2), & \text{on } \{\omega_5, \omega_6, \omega_7, \omega_8\}.
\end{align*}
\]

For \( t = 2 \),

\[
\begin{align*}
1.12 h_0(2) + 2.29 h_1(2) &= 1.12 h_0(3) + 2.29 h_1(3), & \text{on } \{\omega_1, \omega_2\} \\
1.12 h_0(2) + 1.97 h_1(2) &= 1.12 h_0(3) + 1.97 h_1(3), & \text{on } \{\omega_3, \omega_4\} \text{ or } \{\omega_5, \omega_6\} \\
1.12 h_0(2) + 1.69 h_1(2) &= 1.12 h_0(3) + 1.69 h_1(3), & \text{on } \{\omega_7, \omega_8\}. 
\end{align*}
\]

In general, there are many trading strategies that satisfy the specified self-financing conditions.

### 2.2 What is a fair price?

Suppose at \( t = 0 \) you want to evaluate a contract, called a call option, that involves the future stock price: at \( T = 3 \), you have the option of either buying the stock for $2.05 or not buying it. The call option assures you a “no loss” outcome at \( T = 3 \), i.e. your payoff would be \((S(3) - 2.05)^+\). Thus the option should bear a fair price (or called the value of the option) at \( t = 0 \). What should the fair price be?
To answer the question, we use the backward induction: first consider a one-step evolution at the upper right corner of the binomial tree; then extend the result to the entire tree.

Starting from $S(2) = 2.29$, a portfolio $(h_0(3), h_1(3))$ has the value $1.06^2 h_0(3) + 2.29 h_1(3)$, then becomes either $1.06^3 h_0(3) + 2.45 h_1(3)$ or $1.06^3 h_0(3) + 2.11 h_1(3)$ at $T = 3$. If we set

$$\begin{cases} 1.06^3 h_0(3) + 2.45 h_1(3) = 2.45 - 2.05 \\ 1.06^3 h_0(3) + 2.11 h_1(3) = 2.11 - 2.05 \end{cases} (2.1)$$

then the solution $h_0(3) = -1.723$, $h_1(3) = 1$ specifies an investment strategy at $S(2) = 2.29$ that leads to the same payoff as the option, no matter what the outcome of $S(3)$ may be. The value of this (one-step) portfolio, $-1.723 \cdot 1.06^2 + 1 \cdot 2.29 = 0.36$, can be taken as a fair price of the option at $t = 2$ with $S(2) = 2.29$. The following arbitrage argument explains why, provided we assume that any opportunity to make a riskless profit (called an arbitrage opportunity) is ruled out.

Denote the option price by $P$. If $P < 0.36$, then at $t = 2$ a clever investor can buy the option for $P$ and in the meantime follow the strategy $-h_0(3) = 1.723$, $-h_1(3) = -1$. This amounts to short selling one share of stock ($2.29$) (See Hull’s book p48 for how to implement the short-selling.) and deposit $1.723 \cdot 1.06^2$ in the bank. At $T = 3$, the amount collected from the option is exactly what is needed to settle the obligation associated with the portfolio. Hence the investor could lock into a riskless profit of $0.36 - P$. On the other hand, if $P > 0.36$, then the investor would sell short the option for $P$ and follow the strategy $h_0(3) = -1.723$, $h_1(3) = 1$, i.e. to borrow $1.723 \cdot 1.06^2$ from the bank and buy one share of stock ($2.29$). At $T = 3$, the value of the portfolio matches the exact obligation with the option in every possible state of nature. This has the investor lock into a riskless profit of $P - 0.36$.

Therefore, the fair price of the option (or the value of the option) at $S(2) = 2.29$ is $0.36$; By the same token, values of the option at two other $S(2)$ can be derived; Moving one step back, values of the option at two $S(1)$ can be specified; Finally, we end up with the value of the option at $S(0)$. The calculation yields Figure 2.2, a binomial tree for option values \{V(t), t = 0, 1, 2, 3\} (Check it yourself).

In particular, the fair price of the call option at $t = 0$ is $0.28$.

A by-product is Figure 2.3, a portfolio binomial tree. The pair in each box represents the $(h_0, h_1)$ that applies to the two branches connecting the box and its two “descendants”. For example, $h_0(2) = -0.27$ and $h_1(2) = 0.179$ indicate that at $t = 1$, the amount $0.27 \cdot 1.06$ is borrowed from the bank and $0.179$ shares of stocks is bought with the unit price $1.84$. This portfolio is held until the next transaction at $t = 2$. 

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Figure 2.2: Option value tree
Figure 2.3: Portfolio tree
2.3 Risk neutral probabilities

More information can be extracted from this example. Let

\[ q = \frac{1 + r - d}{u - d} = \frac{1.06 - 0.92}{1.07 - 0.92} = \frac{14}{15}, \]  

(2.2)

and denote by \( V(t, k) \) the value of the call option at the location \((t, k)\). Note that the location of each box in this recombining tree is uniquely identified by a pair \((t, k)\) with \( n_t = k \). Then we have

\[ V(t, k) = (1 + r)^{-1} \left[ q \, V(t + 1, k + 1) + (1 - q) \, V(t + 1, k) \right]. \]  

(2.3)

Hence the value in each box is expressed as a discounted weighted average of the values in the two descendent boxes. When the factors \( u, d \) and the rate \( r \) are constants over the tree as in this example, so is the weight \( q \). Such a binomial tree is called a homogeneous tree. Later on we will demonstrate that the method extends to inhomogeneous trees also.

Hence we have two methods to price the call option: one by solving equations like (2.1) thus replicating the portfolio; the other by taking discounted weighted averages like (2.3). The impact is far-reaching. If we define

\[ Q(\omega) = q^{U(\omega)}(1 - q)^{3 - U(\omega)}, \]  

(2.4)

where \( U(\omega) \) represents the total number of up moves in the path \( \omega \), then \( Q \) is a probability measure on \( \Omega \), called a risk neutral probability measure, e.g. \( Q(\omega_3) = q^2(1 - q) = 0.058 \).

It is interesting to notice that the underlying Bernoulli probability \( p \) and the probability measure \( P \) were not relevant in the option pricing. It is the risk neutral probability factor \( q \) and measure \( Q \) that are useful. In general, \( p \neq q \) and \( P \neq Q \). The probability measure \( Q \) is not a part of the model assumptions, but constructed from the market data — stocks and interest rates. In that sense, \( Q \) is an empirical measure.

Exercises:

2.1 Calculate the risk neutral probabilities \( Q(\omega_k), k = 1, \ldots, 8 \).

2.2 Check whether the weighted averages produce the same results in option pricing as in Figure 2.2. Some discrepancies may be due to rounding errors.

2.3 If we change the interest rate from 6\% to 8\%, what would happen? Can you still carry out all the calculation? Why?
Chapter 3

Arbitrage and Risk Neutral Probability Measures

Several important concepts were illustrated in the example in Lecture 2:

• arbitrage;
• risk neutral probability measures;
• contingent claims such as call options;
• two different ways to price a contingent claim.

Now begin our general studies on these topics. Lecture 3 concerns the equivalence between no arbitrage and the existence of risk neutral probability measures. In Lecture 4, we will demonstrate the valuation of a contingent claim by replicating portfolios or taking conditional expectations with respect to a risk neutral probability measure (or called an equivalent martingale measure).

3.1 Some economic considerations

An arbitrage opportunity is said to exist if there is a self-financing strategy $h$ whose value function satisfies

(a) $V(0) = 0$;
(b) $V(T) \geq 0$;
(c) $P(V(T) > 0) > 0$. 
Although a smart investor may seek and grab such a *riskless* way of making a profit, it would only be a transient opportunity because once more people jump in, the prices of the securities would change and the equilibrium would break down. Hence from the economic standpoint, we assume no arbitrage.

**Example 3.1** In the example in Lecture 2, suppose the constant interest rate is 8%. Then an arbitrage opportunity can be found easily. Just do nothing at $t = 0$ and $t = 1$, and short sell one share of stock at $t = 2$, deposit the proceeds in the bank account. This enables the investor to make a net profit at $T = 3$. (Fill in the detail and convince yourself this strategy is self-financing.)

**Example 3.2** Let the interest rate equal 7%. The situation is similar to but slightly more interesting than Example 3.1. Try to find an arbitrage strategy.

In general, it is not easy to check directly whether an arbitrage opportunity exists. A useful criterion is given via equivalent martingale measures.

Assume the finite sample space $\Omega$, the filtration $\mathcal{F}$ as in Lecture 1. A stochastic process $X = \{X(t), t = 0, 1, \ldots, T\}$ is called a martingale under a probability measure $Q$ on $\Omega$ and with respect to $\mathcal{F}$, if the conditional expectation

$$E_Q (X(t) \mid \mathcal{F}_{t-1}) = X(t-1) \quad \forall \ t = 1, \ldots, T.$$  

Sometimes we call $X$ a $Q$-martingale. See Lawler’s book, Chap. 5 for the basic discussion on conditional expectations and martingale. The main result in Lecture 3 is

**Theorem 3.1** No arbitrage $\iff$ there is a probability measure $Q$ with $Q(\omega) > 0 \ \forall \omega \in \Omega$, such that every discounted price process $S^*_n = \{S^*_n(t), t = 0, 1, \ldots, T\}$ is a $Q$-martingale, $n = 1, \ldots, N$. Such a measure $Q$ is called an equivalent martingale measure (EMM).

Results of this kind are sometimes referred to as fundamental theorems of asset pricing. We follow the approach due to Harrison and Pliska given in their seminal paper (1981, *Stoch. Proc. and Their Appl.* 11, 215-260).

### 3.2 Proof of Theorem 3.1: sufficiency “$\Longrightarrow$”

This is an easy direction. It suffices to verify that $\{G^*(t)\}$ is $Q$-martingale [so is $\{V^*(t)\}$ by (1.11)]. Note that by (1.9), for every $t = 1, \ldots, T$, the conditional expectation under $Q$ is

$$E [\Delta G^*(t) \mid \mathcal{F}_{t-1}] = \sum_{n=1}^{N} E [h_n(t) \Delta S^*_n(t) \mid \mathcal{F}_{t-1}] = \sum_{n=1}^{N} h_n(t) E [\Delta S^*_n(t) \mid \mathcal{F}_{t-1}] = 0.$$
The second equality follows from that \( h_n \) is predictable, and the third equality is due to that \( S_n^* \) is a martingale. It is useful to realize that for each \( n \), the process

\[
X_n(t) = \sum_{u=1}^{t} h_n(u) \Delta S_n^*(u)
\]

is also a martingale, as the result of the transform from the martingale \( \{S_n^*(t)\} \) via the predictable process \( h_n \).

### 3.3 Proof of Theorem 3.1: necessity “\( \Rightarrow \)”

A **contingent claim** is a random variable \( Y \) that represents the payoff at time \( T \) from a seller (short position) to a buyer (long position). Recall that the sample space \( \Omega = \{\omega_1, \ldots, \omega_K\} \). Hence the set of possible values \( Y(\omega_1), \ldots, Y(\omega_K) \) of a contingent claim \( Y \) can be considered as an element in \( \mathbb{R}^K \). Let

\[
\mathcal{G} = \{ Y \in \mathbb{R}^K, Y = G^*(T) \text{ for some trading strategy } h \};
\]

\[
\mathcal{A} = \{ Y \in \mathbb{R}^K, Y \geq 0 \text{ and } Y(\omega) > 0 \text{ for some } \omega \in \Omega \};
\]

and

\[
\mathcal{G}^\perp = \{ Z \in \mathbb{R}^K, Y \cdot Z = 0 \forall Y \in \mathcal{G} \}.
\]

Note that \( \mathcal{G} \) is a linear subspace of \( \mathbb{R}^K \) (why?), and \( \mathcal{G}^\perp \) is its orthogonal complement. \( \mathcal{A} \) is the (closed) first quadrant (excluding the origin). No arbitrage implies \( \mathcal{G} \cap \mathcal{A} = \emptyset \). Furthermore, let

\[
\mathcal{W} = \{ Y \in \mathbb{R}^K, Y \geq 0, Y_1 + \ldots + Y_K = 1 \},
\]

which is a closed convex subset of \( \mathcal{A} \). It follows from the **Separating Hyperplane Theorem** that there exists \( \lambda \in \mathcal{G}^\perp \) such that \( \lambda \cdot Y > 0 \) for all \( Y \in \mathcal{W} \). (See Pliska’s book p14 and Duffie’s book p275 for more details.) This implies \( \lambda(\omega) > 0 \) for all \( \omega \in \Omega \). Define a probability measure

\[
Q(\omega) = \frac{\lambda(\omega)}{\sum_{\omega'} \lambda(\omega')}, \quad \omega \in \Omega.
\]

It follows from \( Q \in \mathcal{G}^\perp \) that for any predictable process \( h \),

\[
E_Q \left[ \sum_{n=1}^{N} \sum_{t=1}^{T} h_n(t) \Delta S_n^*(t) \right] = 0.
\]
Hence for every $n$ and any predictable process $h_n$, 

$$E_Q \left[ \sum_{t=1}^{T} h_n(t) \Delta S^*_n(t) \right] = 0.$$ 

This implies that every $S^*_n$ is a $Q$-martingale (why?).

Notes:

(1) The above $\lambda$ is called a state price vector. More on this later.

(2) $Q$ is called an EMM because $Q$ is equivalent to $P$, i.e. for every $\omega \in \Omega$, $Q(\omega) > 0$ if and only if $P(\omega) > 0$. 
Chapter 4

Risk Neutral Valuation of Contingent Claims

A contingent claim $Y$ introduced in Lecture 3 is a contract between a seller and a buyer. Since the seller promises to pay the buyer the amount $Y$ at time $T$, the buyer normally pays some money to the seller at a certain time $t < T$, when they make the agreement.

Q1: What is the appropriate time $t$ value of this contingent claim $Y$? Is it well-defined?

Throughout Lecture 4 we assume no arbitrage. A contingent claim $Y$ is said to be marketable or attainable if there exists a self-financing trading strategy $h$ whose value at $T$ satisfies $V(T) = Y$. In this case, $h$ is said to replicate or generate $Y$.

Q2: Under what conditions on the market, every contingent claim is marketable?

The next two sections answer Q1 and Q2 respectively.

4.1 Law of one price and risk neutral valuation principle

The law of one price is said to hold if there do not exist two trading strategies, say $h$ and $h'$ with corresponding value processes denoted by $\{V(t)\}$ and $\{V'(t)\}$, such that $V(T) = V'(T)$ but $V(t) \neq V'(t)$ for some $t < T$. In other words, if the law of one price holds, then there is no ambiguity about the time $t$ value of any marketable claim at any time $t$.

**Proposition 4.1** No arbitrage $\implies$ the law of one price holds.

**Proof** By Theorem 3.1, there is an EMM $Q$ such that all discounted price processes $S_n^*$, $n = 1, \ldots, N$, thus the discounted value process $\{V^*(t)\}$, are $Q$-martingales. Hence Proposition 4.1 follows (why?).
The converse of Proposition 4.1 is not necessarily true.

**Example 4.1** Revisit Example 3.2. With $r = 0.07$, the equation (2.2) yields $q = 1$. In this case, there is a degenerate probability measure $Q$ defined on $\Omega$ with $Q(\omega_1) = 1$ and $Q(\omega_k) = 0$ for all $k \neq 1$. Note that $Q$ is not an EMM. But we can still use the equation (2.3) to obtain all values. More generally, the law of one price remains true (why?).

**Exercise 4.1** Construct another counterexample in a single period model ($T = 1$).

The following principle is the basis for asset pricing.

**Risk neutral valuation principle:** Assuming no arbitrage, the time $t$ value of a marketable contingent claim $Y$ is equal to $V(t)$, the time $t$ value of the portfolio that replicates $Y$. Moreover,

$$V^*(t) = E_Q [Y/B(T) | \mathcal{F}_t], \quad t = 0, 1, \ldots, T$$

(4.1)

for any EMM $Q$.

**Exercise 4.2** Justify this principle.

### 4.2 Complete markets

The example in Lecture 2 illustrates that for a given contingent claim $Y$, its marketability can be checked by solving a system of linear equations, step by step backwards. Such a tedious procedure is worthwhile because it yields a replicating portfolio when $Y$ is marketable.

Instead of dealing with each individual claim, an alternative approach is to define complete markets: a market is said to be complete if every claim in the market is attainable. A general criterion is:

**Theorem 4.1** An arbitrage-free market is complete $\iff$ there is a unique EMM $Q$.

**Proof**

“$\implies$” Assuming completeness, every contingent claim $Y$ satisfies $Y = V(T)$ for some self-financing strategy $h$. Suppose $Q_1$ and $Q_2$ are two EMMs with the corresponding expectations denoted by $E_1(\cdot)$ and $E_2(\cdot)$.

$$E_1 [Y/B(T)] = E_1 V^*(T) = E_1 V^*(0) = V^*(0),$$

(4.2)

where the second equality is due to that $\{V^*(t)\}$ is a $Q_1$-martingale, and the last equality...
follows from $\mathcal{F}_0 = \{\emptyset, \Omega\}$. By the same token,

$$E_2 [Y/B(T)] = V^*(0). \tag{4.3}$$

Hence $E_1 [Y/B(T)] = E_2 [Y/B(T)]$. This implies $Q_1 = Q_2$ since $Y$ is arbitrary.

“$\Leftarrow$” Assume the market is arbitrage-free but incomplete, and let $\mathcal{C}$ be the set of all marketable contingent claims. Note that $\mathcal{C}$ is a linear subspace of $\mathbb{R}^K$. Thus there exists a contingent claim $Y' \in \mathcal{C}^\perp$, with respect to the inner product $(X, Y) = E_Q (XY)$ on $\mathbb{R}^K$ where $Q$ is an EMM. Define

$$Q'(\omega) = \left[ 1 + \frac{Y'(\omega)}{2 \sup_{\omega \in \Omega} |Y'(\omega)|} \right] Q(\omega), \quad \omega \in \Omega. \tag{4.4}$$

Then

(i) $Q'$ is a probability measure since $E_Q Y' = 0$;

(ii) $Q'(\omega) > 0 \ \forall \omega$ and $Q' \neq Q$;

(iii) $Q'$ is an EMM because for every $n$ and any predictable process $h_n$,

$$E_Q \left[ \sum_{t=1}^T h_n(t) \Delta S_n^*(t) \right] = 0.$$

Exercise 4.3 Construct an example of arbitrage-free but incomplete single period model.
Chapter 5

Binomial Trees: a General Setting

In the next couple of lectures, we will extend the example in Lecture 2 to a general setting — binomial trees, as an important model for a single risky security. It has been extensively used by practitioners in pricing various kinds of derivatives of stocks or bonds. Historically, the model was proposed independently by Cox/Ross/Rubinstein (1979, J. Fin. Econ. 7, 229-263) and Rendleman/Bartter (1979, J. Fin. 34, 1093-1110), although it was often referred to as the CRR model.

5.1 The basic binomial tree model

The evolution of a risky security, say stock, is represented by $S = \{S(t), \ t = 0, 1, \ldots, T\}$. Starting from an initial positive constant price $S(0)$, assume in each time period the stock price either goes up by a factor $u > 1$ with probability $p$, or goes down by a factor $0 < d < 1$ with probability $1 - p$. The moves over time are iid Bernoulli random variables. For each $t$, $S(t) = S(0)u^{n_t}d^{T-n_t}$, where $n_t$ represents the number of up moves up to $t$.

The bank account process $B$ is deterministic with $B(0) = 1$ and a constant interest rate $0 < r < 1$. Hence $B(t) = (1 + r)^t$.

The filtration $\mathcal{F}$ is taken as the one generated by the history of $S$. The sample space $\Omega$ contains $K = 2^T$ different paths. The underlying probability $P$ is defined by $P(\omega) = p^{U(\omega)}(1 - p)^{T-U(\omega)}$, where $U(\omega)$ represents the total number of up moves in the path $\omega$. We assume $0 < p < 1$ so that $P(\omega) > 0 \ \forall \omega \in \Omega$.

As for EMMs, we have the following

**Proposition 5.1** There exists a unique EMM $Q$ $\iff$ $d < 1 + r < u$. In this case,

$$Q(\omega) = q^{U(\omega)}(1 - q)^{T-U(\omega)}, \quad \text{with} \quad q = \frac{1 + r - d}{u - d}.$$  \hspace{1cm} (5.1)
Proof. Let $\xi_t = n_t - n_{t-1}$. Then for every $t$, $S^*(t) = S^*(t-1)(1+r)^{-1}u^{\xi_t}d^{1-\xi_t}$. Therefore,

$$E_Q[S^*(t) \mid \mathcal{F}_{t-1}] = S^*(t-1)$$

$$\iff uQ(\xi_t = 1 \mid n_{t-1}) + d[1 - Q(\xi_t = 1 \mid n_{t-1})] = 1 + r$$

$$\iff Q(\xi_t = 1 \mid n_{t-1}) = \frac{1 + r - d}{u - d},$$

where $Q(\xi_t = 1 \mid n_{t-1})$ denotes the conditional probability (under $Q$) that the next move is up given $n_{t-1}$ up moves up to time $t - 1$. We can denote this (constant) conditional probability by $q$ since it does not depend on $t$ or $n_{t-1}$. This implies that $\xi_1, \ldots, \xi_T$ are iid Bernoulli random variables, and the martingale measure $Q$ is given by (5.1). Note that $0 < Q(\omega) < 1$ for every $\omega$ if and only if $0 < q < 1$ if and only if $d < 1 + r < u$. The above argument also shows such an EMM $Q$ is unique.

**Corollary 5.1** The binomial tree model is a complete market.

### 5.2 Option pricing using binomial trees

A European option is a contingent claim such that the owner of the option may choose (but with no obligation) to exercise it at an expiry or expiration time $T$ and receive the payment $Y$ from the writer of the option. Naturally, the option should be exercised if and only if the payment is positive.

In the simplest case, the contingent claim is expressed as $Y = g(S(T))$ with some function $g$. Using (4.1) in the binomial tree model, the pricing formula for a European option at time $t = 0, 1, \ldots, T - 1$ is given by

$$V(t) = \frac{1}{(1+r)^{T-t}} \sum_{k=0}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k} g(S(t)u^k d^{T-t-k}).$$

(5.2)

Here are some examples.

**Example 5.1** *Call options.* $g(S(T)) = (S(T) - c)^+$ where $c > 0$ is called the exercise price or strike price. A special case was given in Lecture 2. Note that $S(t)u^k d^{T-t-k} - c > 0 \iff k > \frac{\log(c/\langle S(t) u^{T-t} \rangle)}{\log(u/d)}$. Let $k^*$ be the smallest $k$ such that this inequality holds. If $k^* > T - t$, then $V(t) = 0$. If $k^* \leq T - t$, then (5.2) becomes

$$V(t) = S(t) \sum_{k=k^*}^{T-t} \binom{T-t}{k} b^k (1-b)^{T-t-k} - \frac{c}{(1+r)^{T-t}} \sum_{k=k^*}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k},$$

(5.3)

where $b = qu/(1+r) \in (0, 1)$ (why?). The nice thing about this formula is that it involves two sums of $T - t - k^* + 1$ binomial probabilities.
By the put-call parity, derived easily.

or chooses not to exercise the option if $S(T) > c$. A pricing formula similar to (5.3) can be derived easily.

Note: Denote by $c_t$ and $p_t$ respectively, the time $t$ values of the European call and put options with the same expiry $T$ and exercise price $c$. Since $(S(T) - c)^+ - (c - S(T))^+ = S(T) - c$, we have the following put-call parity

$$c_t - p_t = S(t) - \frac{c}{(1+r)^{T-t}}.$$  \hspace{1cm} (5.4)

**Example 5.3 Chooser options.** A chooser option is an agreement that the owner of the option has the right to choose at a fixed decision time $T_0 < T$ whether the option is to be a call or a put with a common exercise price $c$ and remaining time to expiry $T - T_0$. To determine the time $t$ value of the chooser option $(t \leq T_0)$, notice that the payoff at $T$ is

$$(S(T) - c)^+ I_A + (c - S(T))^+ I_{A^c} = (c - S(T))^+ + I_A (S(T) - c),$$

where the event $A = \{c_{T_0} > p_{T_0}\}$, $A^c$ is the complement of $A$, and $I_A$ is the indicator of $A$. By the put-call parity, $c_{T_0} - p_{T_0} = S(T_0) - c (1+r)^{-(T-T_0)}$, which leads to $A = \{S(T_0) > c (1+r)^{-(T-T_0)}\}$. Therefore, the time $T_0$ value of the chooser option is given by

$$E_Q \left[ (c - S(T))^+ + I_A (S(T) - c) \mid \mathcal{F}_{T_0} \right] = p_{T_0} + I_A \left[ S(T_0) - \frac{c}{(1+r)^{T-T_0}} \right] = p_{T_0} + \left[ S(T_0) - \frac{c}{(1+r)^{T-T_0}} \right]^+.$$  

Introducing the notation $C(t, T, c)$ (resp. $P(t, T, c)$) for the time $t$ value of a call (resp. put) option with the expiry $T$ and exercise price $c$, then for any $t = 0, 1, \ldots, T_0$, the time $t$ value $V_{ch}(t)$ of the chooser option can be represented as

$$V_{ch}(t) = P(t, T, c) + C \left(t, T_0, c (1+r)^{-(T-T_0)} \right),$$  \hspace{1cm} (5.5)

or equivalently (why?) as

$$V_{ch}(t) = C(t, T, c) + P \left(t, T_0, c (1+r)^{-(T-T_0)} \right).$$  \hspace{1cm} (5.6)

**Exercise 5.1** Verify (5.5) and (5.6).
Chapter 6

The Black-Scholes Option Pricing Formula

We will show in Lecture 6 that the celebrated Black-Scholes formula in option pricing can be derived from the binomial option pricing formula through an asymptotic argument, provided the parameters in the binomial model are specified appropriately.

Fix $T > 0$, a real number. For a positive integer $n$, partition the interval $[0, T)$ into $[(j - 1)T/n, jT/n), j = 1, \ldots, n$. The previous notation $S(j)$ in the binomial model now represents the stock price at time $jT/n$. Similarly, $B(j)$ represents the bank account at time $jT/n$. Let $r_n = rT/n$ be the interest rate, where $r > 0$ is thought of as the instantaneous rate with the continuous compounding, since $\lim_{n \to \infty} (1 + r_n)^n = e^{rT}$. Let $a_n = \sigma \sqrt{T/n}$ where $\sigma > 0$ is interpreted as the instantaneous volatility. Set the up and down factors by $u_n = e^{a_n}(1 + r_n)$ and $d_n = e^{-a_n}(1 + r_n)$. Note that $d_n < 1$ for sufficiently large $n$.

The risk neutral probability, as $n \to \infty$, has the asymptotic expression

$$q_n = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{1 - e^{-a_n}}{e^{a_n} - e^{-a_n}} = \frac{a_n - \frac{1}{2} a_n^2 + o(a_n^2)}{2a_n + \frac{1}{3} a_n^3 + o(a_n^3)} = \frac{1}{2} - \frac{1}{4} a_n + o(a_n),$$

where the notation $o(\epsilon)$ with $\epsilon > 0$ means $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$.

Recall the iid Bernoulli random variables $\xi_j, j = 1, \ldots, n$ introduced in Lecture 5, with $Q(\xi_j = 1) = q_n$. The stock price at $T$ is represented as

$$S(n) = S(0) u_n^{\xi_1 + \cdots + \xi_n} d_n^{-(\xi_1 + \cdots + \xi_n)}.$$
Hence the value of the put option at time 0 is given by

\[ p_0^{(n)} = (1 + r_n)^{-n} E_Q (c - S(n))^+ = E_Q \left( \frac{c}{(1 + r_n)^n} - S(0) \ e^{Y_n} \right)^+, \quad (6.1) \]

where

\[ Y_n = \sum_{j=1}^{n} Y_{n,j} = \sum_{j=1}^{n} \log \left( \frac{u_{n,j} d_{n,j}^{1-\xi_j}}{1 + r_n} \right). \quad (6.2) \]

Note that for fixed \( n \), \( Y_{n,1}, \ldots, Y_{n,n} \) are iid random variables with

\[ E_Q Y_{n,j} = q_n \log \frac{u_n}{1 + r_n} + (1 - q_n) \log \frac{d_n}{1 + r_n} = \frac{-1}{2} a_n^2 + o(a_n^2), \quad (6.3) \]

\[ E_Q Y_{n,j}^2 = a_n^2, \quad (6.4) \]

and

\[ E_Q |Y_{n,j}|^m = o(a_n^2) \quad \forall \ m = 3, 4, \ldots. \quad (6.5) \]

Using characteristic functions [see the note after (6.8)], it follows that \( Y_n \) converges in distribution to \( N(-\sigma^2 T/2, \sigma^2 T) \) as \( n \to \infty \). It is noteworthy that the family \( \{Y_{n,j}\} \) is a triangular array, hence the asymptotic distribution of \( Y_n \) need not always belong to the Gaussian distribution family. In other words, the argument here goes somewhat beyond the basic form of “Central Limit Theorem”.

Since

\[ |p_0^{(n)} - E_Q \left( c \ e^{-rT} - S(0) \ e^{Y_n} \right)^+| \leq c \ |(1 + r_n)^{-n} - e^{-rT}|, \quad (why?) \quad (6.6) \]

we have

\[
\lim_{n \to \infty} p_0^{(n)} = \lim_{n \to \infty} E_Q \left( c \ e^{-rT} - S(0) \ e^{Y_n} \right)^+ \\
= \int_{-\infty}^{\infty} e^{-x^2/2} \left[ c \ e^{-rT} - S(0) \ \exp \left( -\frac{\sigma^2 T}{2} + \sigma \sqrt{T}z \right) \right]^+ \ dz \\
= c \ e^{-rT} \Phi(-v_2) - S(0) \ \Phi(-v_1),
\]

where \( v_1 = \frac{\log(S(0)/c) + (r + \sigma^2/2) T}{\sigma \sqrt{T}}, \quad v_2 = v_1 - \sigma \sqrt{T} = \frac{\log(S(0)/c) + (r - \sigma^2/2) T}{\sigma \sqrt{T}} \), and \( \Phi \) is the cumulative distribution function of \( N(0, 1) \).
This is the Black-Scholes pricing formula for a European option. We choose to consider put options first since their payoff (or loss) functions are bounded which make the asymptotic argument easier. The following pricing formula for a call option can be derived using put-call parity:

$$\lim_{n \rightarrow \infty} c^{(n)}_0 = S(0) \Phi(v_1) - c e^{-rT} \Phi(v_2).$$

Furthermore, by changing 0 to any $t \in (0, T)$ and $T$ to $T - t$, the same argument goes through, which provides the Black-Scholes formulas for pricing the time $t$ value $C(t, T)$ of a (European) call option:

$$C(t, T) = S(t) \Phi(v_1) - c e^{-r(T-t)} \Phi(v_2),$$

and the time $t$ value $P(t, T)$ of a (European) put option:

$$P(t, T) = c e^{-r(T-t)} \Phi(-v_2) - S(t) \Phi(-v_1),$$

(6.7)

where $v_1 = \frac{\log(S(t)/c) +(r+\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$ and $v_2 = v_1 - \sigma \sqrt{T-t} = \frac{\log(S(t)/c) +(r-\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$.

**Note:** To verify that $Y_n$ converges in distribution to $N(-\sigma^2 T/2, \sigma^2 T)$ as $n \rightarrow \infty$, consider the characteristic function $E_Q e^{i w Y_n}$ of $Y_n$ where $w \in \mathbb{R}$ and $i = \sqrt{-1}$ (imaginary unit in complex analysis). Following the fact that $Y_{n,1}, \ldots, Y_{n,n}$ are iid, and (6.3) — (6.5), we have the Taylor expansion

$$E_Q e^{i w Y_n} = \prod_{j=1}^{n} E_Q e^{i w Y_{n,j}}$$

$$= \left( 1 + iw E_Q Y_{n,j} - \frac{w^2}{2} E_Q Y_{n,j}^2 - \frac{i \theta^3}{3!} E_Q Y_{n,j}^3 \right)^n$$

$$\rightarrow \exp (-iw\sigma^2 T/2 - w^2\sigma^2 T/2)$$

as $n \rightarrow \infty$, where $\theta$ satisfies $|\theta| \leq |w|$. Note that $\exp (-iw\sigma^2 T/2 - w^2\sigma^2 T/2)$ is just the characteristic function of $N(-\sigma^2 T/2, \sigma^2 T)$.

**Exercise 6.1** Derive the formula (6.7).
Chapter 7

American Options as Optimal Stopping Problems

An American option is a contract between two parties made at a certain time $t$ such that the buyer of the contract has the right, but not the obligation, to exercise the option at any time $\tau$ with $t \leq \tau \leq T$. If the option is exercised at $\tau$, then the seller pays the buyer an amount $Y(\tau) \geq 0$. For instance, $Y(\tau) = (S(\tau) - c)^+$ for an American call option and $Y(\tau) = (c - S(\tau))^+$ for an American put option based on a single stock. One can identify an American option by its payoff process $Y^A = \{Y(t), t = 0, 1, \ldots, T\}$. American options enjoy the additional flexibility — possibility of exercising earlier than $T$ — compared to their European option counterparts. What is the value $V^A(t)$ of an American option?

7.1 A special case: “American = European”

Since the holder of an American option can always choose not to exercise the option until time $T$, $V^A(t) \geq V(t)$ where $V(t)$ is the time $t$ value of the European option with the payoff $Y = Y(T)$. Nevertheless, there are situations where the two value processes coincide.

**Proposition 7.1** Consider an American option $Y^A$ and the corresponding European option with time $T$ value $Y = Y(T)$. If $V(t) \geq Y(t)$ for all $t$, then $V(t) = V^A(t)$ for all $t$, and it is optimal to wait until time $T$ to exercise.

**Proof** For the holder of an American option, exercising at $t$ only ends up with payoff $Y(t)$, while selling the corresponding European option (or shorting the portfolio which replicates the European option) would guarantee you a time $t$ payoff $V(t)$. Hence the option should not be exercised at $t$. Since $t$ is arbitrary, it is optimal to wait until $T$ to decide whether to exercise.
Consider the American call option with $Y(t) = (S(t) - c)^+$ at each $t$ where $c = 2.05$ in the example given in Lecture 2. Proposition 7.1 applies to this case. See Figure 7.1.

**Note:** The fact that an American call option is equivalent to its European counterpart is due to its special probability structure. We will discuss this later.

### 7.2 Optimal stopping

Section 7.1 is not a typical case. You may check the American put option in the same example. For instance, when $S(2) = 1.69$, an immediate exercise gives you the payoff $2.05 - 1.69 = 0.36$, compared to the value of the corresponding European put:

$$1.06^{-1} \left[ \frac{14}{15} (2.05 - 1.81) + \frac{1}{15} (2.05 - 1.56) \right] = 0.24 < 0.36.$$ Hence postponing the exercise decision until $T$ is suboptimal (why?).

To study when it is optimal to exercise an American option and evaluate the option, we need to introduce supermartingales, submartingales and stopping times.

A stochastic process $X = \{X(t), t = 0, 1, \ldots, T\}$ is called a $Q$-supermartingale under a probability measure $Q$ on $\Omega$ and with respect to $\mathcal{F}$, if the conditional expectation

$$E_Q(X(t) | \mathcal{F}_{t-1}) \leq X(t-1) \quad \forall \ t = 1, \ldots, T;$$

On the other hand, $X$ is called $Q$-submartingale if

$$E_Q(X(t) | \mathcal{F}_{t-1}) \geq X(t-1) \quad \forall \ t = 1, \ldots, T.$$ All martingales are both supermartingales and submartingales, but not vice versa. Recall that the discounted value process of a European option is a $Q$-martingale under an EMM $Q$. It turns out that the discounted value process of an American option is a $Q$-supermartingale.

A stopping time $\tau$ is a random variable taking values in the set $\{0, 1, \ldots, T; \infty\}$ such that for every $t \leq T$, the event $\{\tau = t\} \in \mathcal{F}_t$, i.e. the information on whether $\{\tau = t\}$ occurs is available at time $t$. As a simple example, suppose the stock price $S(0) = 2$, then $\tau_1 = \min\{t: S(t) > 2.1\}$ is a stopping time, but $\tau_2 = \max\{t: S(t) > 2.1\}$ is not a stopping time. We allow stopping times to take the value $\infty$ in order to represent some events of interest that never occur up to time $T$.

An American option $Y^A$ is said to be marketable if for every stopping time $\tau \leq T$ the contingent claim $Y(\tau)$ can be replicated. Here is a basic result for American option pricing.

**Theorem 7.1** For an EMM $Q$, define a stochastic process $Z = \{Z(t), \ t = 0, 1, \ldots, T\}$
Figure 7.1: Exercise at $t$ or $T$?
iteratively via the dynamic programming equations

\[
\begin{aligned}
Z(T) &= Y(T) \\
Z(t) &= \max \{ Y(t), \ E_Q \ [Z(t+1)B(t)/B(t+1) \ | \ \mathcal{F}_t] \} , \quad t \leq T - 1.
\end{aligned}
\]  

(7.1)

Then

(a) For each \( t \),

\[
Z(t) = \max \tau \ E_Q \ [Y(\tau)B(t)/B(\tau) \ | \ \mathcal{F}_t],
\]

where the maximum is over all stopping times \( t \leq \tau \leq T \).

(b) The maximum on the RHS of (7.2) is attained by the stopping time

\[
\tau(t) = \min \{ t' \geq t : \ Z(t') = Y(t') \}.
\]

(7.3)

(c) The discounted version \( Z^* \) of \( Z \) is the smallest \( Q \)-supermartingale satisfying

\[
Z(t) \geq Y(t) \quad \forall t.
\]

(7.4)

(Z is called the Snell envelope of \( Y^A \).)

(d) For a marketable American option \( Y^A \), its value process is given by

\[
V^A(t) = Z(t) \quad \forall t,
\]

(7.5)

and the optimal (early) exercise strategy at time \( t \) is given by the stopping time \( \tau(t) \).

Proof

For (a) and (b), use backward induction. (7.2) and (7.3) clearly hold for \( t = T \). Suppose (7.2) holds for \( t \), then

\[
\begin{aligned}
Z(t - 1) &= \max \{ Y(t - 1), \ E_Q[Z(t)B(t - 1)/B(t) \ | \ \mathcal{F}_{t-1}] \} \\
&= \max \{ Y(t - 1), \ E_Q[\max_{\tau \geq t} E_Q[Y(\tau)B(t)/B(\tau) \ | \ \mathcal{F}_t] B(t - 1)/B(t) \ | \ \mathcal{F}_{t-1}] \} \\
&\geq \max \{ Y(t - 1), \ E_Q[E_Q[Y(\tau)B(t)/B(\tau) \ | \ \mathcal{F}_t] B(t - 1)/B(t) \ | \ \mathcal{F}_{t-1}] \} \\
&= \max \{ Y(t - 1), \ E_Q[Y(\tau)B(t - 1)/B(\tau) \ | \ \mathcal{F}_{t-1}] \}
\end{aligned}
\]

for any stopping time \( \tau \geq t \). Hence

\[
\begin{aligned}
Z(t - 1) &\geq \max \{ Y(t - 1), \ \max_{\tau \geq t} E_Q[Y(\tau)B(t - 1)/B(\tau) \ | \ \mathcal{F}_{t-1}] \} \\
&\geq \max_{\tau \geq t} E_Q[Y(\tau)B(t - 1)/B(\tau) \ | \ \mathcal{F}_{t-1}] \quad \text{(why?)}.
\end{aligned}
\]
On the other hand, assuming (7.3) for \( t \) leads to

\[
Z(t - 1) = \max \{ Y(t - 1), E_Q[Z(t)B(t - 1)/B(t) \mid \mathcal{F}_{t-1}] \} = \max \{ Y(t - 1), E_Q\{ E_Q[Y(\tau(t))B(\tau(t)) / B(\tau(t)) \mid \mathcal{F}_t] B(t - 1)/B(t) \mid \mathcal{F}_{t-1}] \} = \max \{ Y(t - 1), E_Q[Y(\tau(t))B(t - 1)/B(\tau(t)) \mid \mathcal{F}_{t-1}] \} = E_Q[Y(\tau(t - 1))B(t - 1)/B(\tau(t - 1)) \mid \mathcal{F}_{t-1}] \quad \text{(why?)} \leq \max_{\tau \geq t-1} E_Q[Y(\tau)B(t - 1)/B(\tau) \mid \mathcal{F}_{t-1}].
\]

Therefore, (7.2) and (7.3) have been verified for \( t - 1 \).

For (c), it follows from (7.1) that \( Z^* \) is a \( Q \)-supermartingale and \( Z(t) \geq Y(t) \) for all \( t \). Suppose \( U \) is another process such that \( U^* \) is a \( Q \)-supermartingale and \( U(t) \geq Y(t) \) for all \( t \). Then

\[
U(t - 1) \geq \max \{ Y(t - 1), E_Q[U(t)B(t - 1)/B(t) \mid \mathcal{F}_{t-1}] \} \quad t = 1, \ldots, T - 1. \quad (7.6)
\]

Starting from \( U(T) \geq Y(T) = Z(T) \) and working backwards iteratively in (7.6) and (7.1) will lead to \( U(t) \geq Z(t) \) for all \( t \).

For (d), we use an arbitrage argument (or called hedging) as follows.

Suppose \( V^A(t) > Z(t) \). Then one can sell the option for \( V^A(t) \) and take a portfolio replicating \( Y(\tau(t)) \) at the cost \( Z(t) \) and invest \( V^A(t) - Z(t) \) in the bank account. Later, if the buyer exercises the option at some time \( \tau \leq \tau(t) \), you liquidate the portfolio, collect \( Z(\tau) \) and pay the buyer \( Y(\tau) \). These transactions guarantee you a positive profit. On the other hand, if the buyer does not exercise by \( \xi = \tau(t) < T \), then you repeat this process: take a portfolio replicating \( Y(\tau(\xi)) \) at the cost \( E_Q[Z(\xi + 1)B(\xi)/B(\xi + 1) \mid \mathcal{F}_\xi] \), which is at most \( Z(\xi) = Y(\xi) \) by (7.1). As before, if the buyer exercises at some time \( \tau \leq \tau(\xi) \), then the value of the portfolio will be enough to cover the payoff \( Y(\tau) \). If the buyer does not exercise by \( \tau(\xi) \), then you repeat the process once again, etc. The basic fact is that you always have enough money in the portfolio to cover the needed payoff, and you overall profit will be at least \( V^A(t) - Z(t) > 0 \).

For the opposite case \( V^A(t) < Z(t) \), you can reverse the strategy: buy the option for \( V^A(t) \), take the negative of the previous portfolio, collect \( Z(t) \) and invest the difference \( Z(t) - V^A(t) \) in the bank account. Later you exercise the option at time \( \tau(t) \) and liquidate the replicating portfolio at the same time. Since \( V(\tau(t)) = Y(\tau(t)) \), the amount you collect from the option seller is exactly equal to your liability on the portfolio. In the mean time, you have \( [Z(t) - V^A(t)] B(\tau(t))/B(t) > 0 \) in your bank account.

Therefore, there would be an arbitrage opportunity if \( V^A(t) \neq Z(t) \). Moreover, (7.3) specifies an optimal exercise strategy for the American option buyer, because other strategies
would run the possible risk of exercising when $Z(\tau) > Y(\tau)$ at some time $\tau$. In that case, the buyer would sacrifice the amount $Z(\tau) - Y(\tau) > 0$.

We have thus far completed the proof of Theorem 7.1.

**Exercise 7.1** Construct a binomial tree for the American put option values and the optimal exercise times in the example given in Lecture 2.
Chapter 8

More on Valuation of American Options

8.1 “American calls = European calls”

Yes, the above quote is really true, i.e. American calls have the same values as their European counterparts in the simple set-up given in Lecture 7, thus there should be no earlier exercises. This is because \( \{Y^*(\tau)\} \) is a submartingale.

**Proposition 8.1** If \( \{Y^*(\tau)\} \) is a \( Q \)-submartingale for a marketable American option \( Y^A \), then for every \( t = 0, 1, \ldots, T \), the optimal exercise strategy is just \( \tau(t) = T \), and \( V^A(t) = V(t) \), where \( V(t) \) is the time \( t \) value of the European option with terminal payoff \( Y(T) \).

**Proof**

\[
V^A(t) = E_Q[Y(\tau(t))B(t)/B(\tau(t)) \mid \mathcal{F}_t] = \sum_{s=t}^{T} E_Q \left[ Y(s)B(t)/B(s) I_{\{\tau(t) = s\}} \mid \mathcal{F}_t \right] \leq \sum_{s=t}^{T} E_Q \left\{ E_Q \left[ Y(T)B(t)/B(T) I_{\{\tau(t) = s\}} \mid \mathcal{F}_s \right] \mid \mathcal{F}_t \right\} = E_Q[Y(T)B(t)/B(T) \mid \mathcal{F}_t] = V(t).
\]

**Corollary 8.1** In the set-up given in Lecture 7, there should be no exercise earlier than \( T \) for an American call option.
Proof. To check \( \{Y^*(t)\} \) is a submartingale, note that for every \( t = 1, \ldots, T, \)
\[
E_Q[(S^*(t) - c / B(t))^+ \mid \mathcal{F}_{t-1}] \\
\geq E_Q[S^*(t) - c / B(t) \mid \mathcal{F}_{t-1}] \\
= S^*(t-1) - c E_Q[1/B(t) \mid \mathcal{F}_{t-1}] \\
\geq S^*(t-1) - c / B(t-1).
\]
Since \( E_Q[(S^*(t) - c / B(t))^+ \mid \mathcal{F}_{t-1}] \geq 0 \), we have \( E_Q[(S^*(t) - c / B(t))^+ \mid \mathcal{F}_{t-1}] \geq [S^*(t-1) - c / B(t-1)]^+. \)

Note: It is a crucial part of the proof of Corollary 8.1 that \( S^* \) is a \( Q \)-martingale, which is not the case in the next section.

8.2 Options on a dividend-paying stock

In a more realistic market, some dividend-paying stocks may issue cash payments, called dividends, to shareholders on a periodic basis. A dividend is referred to as the reduction in the stock price on the ex-dividend date. There may be several ex-dividend dates in a stock, and possibly many different forms of dividends. For illustration, we consider the binomial tree in Lecture 2 again, where \( T = 3 \) is the only ex-dividend date and the dividend is issued as a constant yield \( \lambda \) of the stock. This means the shareholder will receive a dividend payment at \( T \) which amounts to either \( \lambda u S(T-1) \) or \( \lambda d S(T-1) \) according to the stock fluctuation. In the meantime, the ex-dividend stock price at time \( T \) will be either \( (1 - \lambda) u S(T-1) \) or \( (1 - \lambda) d S(T-1) \). This corresponds to the traditional assumption that the stock price declines on the ex-dividend date by the dividend amount. One can easily complete this modified binomial tree.

Various options (calls, puts, and others; European or American) on dividend-paying stocks can be priced virtually in the same way as before, except that the exercise payoff is identified as \( [(1 - \lambda) u S(T-1) - c]^+ \) or \( [(1 - \lambda) d S(T-1) - c]^+ \) for calls, etc. In this situation, the methodology using binomial trees enjoys its great flexibility.

This example also shows that on a dividend-paying stock, an American call need not have the same value as its European counterpart. Corollary 8.1 does not hold for American call options on dividend-paying stocks. To see this, consider the inequality
\[
(1 + r)[S(T-1) - c]^+ \\
> \ q \ [(1 - \lambda) u S(T-1) - c]^+ + (1 - q) \ [(1 - \lambda) d S(T-1) - c]^+, \quad (8.1)
\]
which amounts to \( 1.06 \ (2.29 - 2.05) > \frac{14}{15}[(1 - \lambda) 2.45 - 2.05]^+ + \frac{1}{15}[(1 - \lambda) 2.11 - 2.05]^+ \) when \( S(2) = 2.29 \), thus \( \lambda > 0.05 \) leads to exercising at \( t = 2. \)
Another simple form of dividend is “known dollar dividend” (see Hull’s book p354).
Chapter 9

Return and Risk

One of the important problems in modern financial economics is the quantification of the trade-off between risk and expected return. Although common sense suggests that risky investments such as stocks will generally yield higher returns than riskless investments, it was the development of the Capital Asset Pricing Model (CAPM) that enables economists to quantify risk and the reward for bearing it. The CAPM implies that the expected return of an asset must be linearly related to the covariance of its return with the return of the market portfolio.

We will introduce the concepts of return and risk, then discuss some basic versions of CAPM.

9.1 Return processes

In the basic set-up in Lecture 1, assume a constant interest rate \( r(t) \equiv r \) for simplicity. For each \( n = 1, \ldots, N \), define the corresponding return process \( R_n = \{ R_n(t) \} \) by \( R_n(0) = 0 \), and for \( t = 1, \ldots, T \),

\[
R_n(t) = \begin{cases} 
\Delta S_n(t)/S_n(t-1), & \text{if } S_n(t-1) > 0 \\
0, & \text{if } S_n(t-1) = 0.
\end{cases}
\] (9.1)

Also, \( R_0(t) \equiv r \) for the bank account.

It is easy to see that for \( t = 1, \ldots, T \),

\[
S_n(t) = S_n(0) + \sum_{v=1}^{t} S_n(v-1)R_n(v),
\] (9.2)
and

\[ S_n(t) = S_n(0) \prod_{v=1}^{t} [1 + R_n(v)]. \]  

(9.3)

What is the relation between \( R_n \) and \( R^*_n \), which is the return process corresponding to \( S^*_n \)? Note that

\[
\Delta S^*_n(t) = S_n(t)/B(t) - S^*_n(t-1) = \frac{S_n(t-1) [1 + R_n(t)]}{(1 + r)^t} - S^*_n(t-1)
\]

and

\[
\Delta S^*_n(t) = S^*_n(t-1) R^*_n(t) \]

by definition. Hence

\[
R^*_n(t) = \frac{R_n(t) - r}{1 + r},
\]

(9.4)

Now we define the return process \( R = \{ R(t) \} \) of a portfolio \( h \) as the return corresponding the value process \( V \) for \( h \): \( R(0) = 0 \) and for \( t = 1, \ldots, T \),

\[
R(t) = \begin{cases} 
\Delta V(t)/V(t-1), & \text{if } V(t-1) > 0 \\
0, & \text{if } V(t-1) \leq 0.
\end{cases}
\]

(9.5)

If we let \( R^* \) be the return process corresponding to \( V^* \), i.e. defined as in (9.5) with \( V \) replaced by \( V^* \), then

\[
R^*(t) = \frac{R(t) - r}{1 + r}.
\]

(9.6)

Furthermore, since \( V(t) - V(t-1) = h_0(t) \Delta B(t) + \sum_{n=1}^{N} h_n(t) \Delta S_n(t) \), we have

\[
R(t) = \left[ \frac{h_0(t)(1+r)^{t-1}}{V(t-1)} \right] r + \sum_{n=1}^{N} \left[ \frac{h_n(t)S_n(t-1)}{V(t-1)} \right] R_n(t),
\]

(9.7)

which expresses the return for the portfolio as a linear combination of the returns for the individual securities. In this expression, \( \frac{h_n(t)S_n(t-1)}{V(t-1)} \) represents the fraction of the investor’s wealth invested in security \( n \) at time \( t-1 \) and about to be carried forward to time \( t \).

### 9.2 Risk premium (single period)

The original version of CAPM is a single period model. To illustrate the basic idea, we introduce the risk premium in a single period model \((T = 1)\) first, and then extend it to the multi-period case later.
For \( \omega \in \Omega \), the ratio \( L(\omega) = Q(\omega)/P(\omega) \) is called the state price density (or state price deflator), where \( P \) and \( Q \) denote the underlying probability measure and an EMM respectively (i.e. we assume no arbitrage). Write \( R_n(1) = R_n \) for each \( n \), and \( R(1) = R \). Note that

\[
E_Q \left( \frac{R_n - r}{1 + r} \right) = 0.
\]  
(9.8)

Hence the covariance

\[
\text{cov}(R_n, L) = E_P(R_n L) - E_P R_n E_P L = E_Q R_n - E_P R_n = r - E_P R_n.
\]

Thus the difference \( E_P R_n - r \), called the risk premium for security \( n \), is expressed as

\[
E_P R_n - r = - \text{cov}(R_n, L).
\]  
(9.9)

Normally this is positive since investors usually believe that the expected returns of risky securities are higher than the riskless return \( r \).

Following (9.7) with \( t = 1 \), we have

\[
E_P R - r = - \text{cov}(R, L).
\]  
(9.10)

Consider a class of marketable contingent claims of the form \( a + bL \) where \( a \) and \( b \) are two constants with \( b \neq 0 \). Any such a claim is perfectly correlated with the state price deflator \( L \). Suppose \( h' \) is the strategy replicating \( a + bL \) whose value and return are denoted by \( V' \) and \( R' \) respectively. Then \( a + bL = V'(1) = V'(0)(1 + R') \). Thus

\[
\text{cov}(R, L) = \frac{V'(0)}{b} \text{cov}(R, R').
\]

Hence (9.10) becomes

\[
E_P R - r = - \frac{V'(0)}{b} \text{cov}(R, R').
\]  
(9.11)

Setting \( h = h' \) (note that \( h \) is an arbitrary portfolio) turns (9.11) to

\[
E_P R' - r = - \frac{V'(0)}{b} \text{var}(R').
\]  
(9.12)

Using this to substitute for \( \frac{V'(0)}{b} \) in (9.11), we obtain

**Proposition 9.1** Let \( R' \) be the return of a marketable contingent claim \( a + bL \) and \( R \) the return of an arbitrary portfolio \( h \). Then

\[
E_P R - r = \beta_h \left( E_P R' - r \right),
\]  
(9.13)
where $\beta_h = \frac{\text{cov}(R, R')}{\text{var}(R')}$ is called the beta of the trading strategy $h$ with respect to the trading strategy $h'$ that replicates $a + bL$.

(9.13) is referred to as a state price beta model, showing that the risk premium of a portfolio is proportional to the risk premium of a portfolio perfectly correlated with a state price deflator, where the proportional constant is the associated regression coefficient.

**Note:** Recall that the unique best linear predictor (to minimize the mean squared error) of $Y$ based on $X$ is given by $\hat{Y} = EY + \beta (X - EX)$ where $\beta = \text{cov}(X, Y)/\text{var}(X)$. 


Chapter 10

Optimal Portfolios

In the next couple of lectures, we will study optimization of portfolios, its connection with CAPM. It is noticeable that while only an EMM $Q$ is relevant in option pricing, both the underlying probability measure $P$ and the EMM $Q$ play an important role in this part of the studies.

10.1 Optimal portfolios

The objective in a basic optimal portfolio problem is to maximize the expected utility of time $T$ wealth, assuming there is no consumption before $T$.

Let $U(w)$ represent the utility of wealth $w$ at time $T$ and assume the function $U$ is differentiable, concave and strictly increasing. Here is a basic problem:

(P1) Find a self-financing strategy $h$ with $V(0) = w_0$ (a given initial wealth) to maximize $E U(V(T))$, where $E(\cdot)$ denotes the expectation with respect to $P$.

The following problem is equivalent to (P1):

(P2) Find predictable processes $h_1, \ldots, h_N$ to maximize $E U(B(T) [w_0 + G^*(T)])$.

The reason that (P2) is equivalent to (P1) is this: for a solution $(h_1, \ldots, h_N)$, it is easy to choose $h_0$ such that $h = (h_0, h_1, \ldots, h_n)$ is self-financing with $V(0) = w_0$.

Before solving (P2), notice that if there exists an arbitrage opportunity, then there cannot exist an optimal portfolio as a solution to (P1) or (P2). In other words, we have
Proposition 10.1 If $h$ is a solution to (P1) or (P2), then there exists an EMM given by

$$Q(\omega) = \frac{P(\omega) \ B(T;\omega) \ U'(V(T;\omega))}{E[B(T) \ U'(V(T))]}, \quad \omega \in \Omega$$

where $U'(V(T)) = \frac{dU(w)}{dw} \bigg|_{w=V(T)}$.

Proof For any $t$, $n$ and event $A \in \mathcal{F}_{t-1}$, $h_n(t;\omega)$ specifies the optimal trading position in security $n$ carried forward from time $t-1$ to time $t$ when $\omega \in A$, with other components of $h$ fixed. The first-order necessary condition for this restricted optimality is

$$\sum_{\omega \in A} P(\omega) \ U'(B(T;\omega) \ [w_0 + G^*(-T;\omega)]) \ B(T;\omega) \Delta S_n^*(t;\omega) = 0,$$

thus $E \{U'(B(T) \ [w_0 + G^*(T)]) \ B(T)\Delta S_n^*(t) \mid \mathcal{F}_{t-1}\} = 0$, which implies $E_Q[\Delta S_n^*(t) \mid \mathcal{F}_{t-1}] = 0$ by (10.1).

10.2 Computation via dynamic programming

The computation involved in solving (P2) can be intensive, especially when $T$, $N$ and $K$ are large. The usual calculus-based optimization procedure would be intractable. A more efficient method is dynamic programming by which a multiperiod decision problem can be reduced to a sequence of single period problems, working backwards in time. To implement this, let $U(t,w)$ be the maximum expected utility of time $T$ wealth given the current time $t$, the current wealth $w$, and the history $\mathcal{F}_t$.

Starting from $U(T,w) = U(w)$, it follows that for $t = 0,1,\ldots,T-1$,

$$U(t,w) = \max_{h} \mathbb{E} \left\{ U(t+1,B(t+1) \left[ w/B(t) + \sum_{n=1}^{N} h_n(t+1)\Delta S_n^*(t+1) \right] \mid \mathcal{F}_t \right\},$$

where the maximum is over $\mathcal{F}_t$-measurable random variables $h_1(t+1),\ldots,h_N(t+1)$; and $h_0(t+1)$ is chosen in a self-financing manner.

To illustrate this, consider the example in Lecture 2 in which we let $p = 0.4$ and set the utility $U(w) = 1 - e^{-w}$. The RHS of (10.2) becomes

$$\max_{h} \mathbb{E} \left\{ 1 - \exp [-1.06w - h \ (S(3) - 1.06 \ S(2))] \mid S(2) = 2.29 \right\}$$

$$= \max_{h} \{ 1 - \exp(-1.06w) \ g(h) \}$$

$$= 1 - \exp(-1.06w) \ \min_{h} g(h),$$
where

\[ g(h) = 0.4 \exp(-h(2.45 - 1.06 \cdot 2.29)) + 0.6 \exp(-h(2.11 - 1.06 \cdot 2.29)), \]

and \( h \in \mathbb{R} \) (not the notation for the entire portfolio). Setting the derivative of \( g(h) \) with respect to \( h \) equal to zero will lead to \( h = -8.96376 \). Plugging this back into \( g(h) \) will yield the maximal expected value \( U(2, w) \) conditioning on \( S(2) = 2.29 \).

**Exercise 10.1** Complete the dynamic programming procedure for this example. Think about how the value of \( w \) affects the result.
Chapter 11

Optimization via EMMs

In this lecture, we present an alternative method for solving (P1) or (P2) in Lecture 10, using an EMM $Q$. Later in this course, we will demonstrate this useful approach in some other problems, such as model calibration.

11.1 Basic approach

Assume the market is complete and let $C_{w_0}$ be the set of time $T$ contingent claims that can be generated by some self-financing strategy starting with initial wealth $w_0$. We can also write

$$C_{w_0} = \{ Y \in \mathbb{R}^K : E_Q \left[ Y/B(T) \right] = w_0 \}.$$

The following problem is obviously equivalent to (P1) or (P2):

(P3) There are two parts:

(i) Maximize $E U(Y)$ subject to $Y \in C_{w_0}$;

(ii) Find a self-financing strategy $h$ to replicate the solution $\hat{Y}$ of (i).

Since we are familiar with Part (ii) already, we now consider Part (i) by introducing a Lagrange multiplier $\lambda$ and maximizing $E U(Y) - \lambda E_Q \left[ Y/B(T) \right]$. Using the state price density, this problem can be rewritten as maximizing the objective function

$$E \left[ U(Y) - \lambda L Y/B(T) \right] = \sum_{\omega} P(\omega) \left[ U(Y(\omega)) - \lambda L(\omega) Y(\omega)/B(T; \omega) \right].$$

The necessary condition is that for each $\omega \in \Omega$,

$$U'(Y(\omega)) = \lambda L(\omega)/B(T; \omega),$$

(11.1)
thus
\[ Y = (U')^{-1}(\lambda L / B(T)), \]
(11.2)
where \((U')^{-1}\) represents the inverse function of \(U'\). Moreover, the correct value of \(\lambda\) is chosen to satisfy
\[ E_Q [(U')^{-1}(\lambda L / B(T)) / B(T)] = w_0. \]
(11.3)

### 11.2 Examples

The following examples will show that to compute the optimal expected utility \((P3)\) Part (i), the EMM approach is much more efficient than the dynamic programming introduced in Lecture 10. This is because the optimal portfolio has to be found before calculating the optimal expected utility when using dynamic programming. While using the EMM “Divide-and-Conquer” method, the optimal expected utility is identified via an intermediate variable \(Y\) which has a much smaller dimensionality compared to \(h\). Although finding the optimal portfolio \(h\) \((P3)\) Part (ii) still involves a significant amount of calculation, it is in a familiar territory — asset pricing. We will derive the optimal solution \(Y\) in each example, then find the optimal expected utility and the optimal portfolio in one example, and leave the other examples to the reader.

**Example 11.1** For the exponential utility \(U(w) = 1 - e^{-w}\), we have \((U')^{-1}(z) = -\log z, z > 0\). Hence (11.2) and (11.3) imply

\[ Y = -\log[L/B(T)] + [E(L/B(T))]^{-1} \{w_0 + E[L/B(T) \log(L/B(T))]\}. \]
(11.4)

**Example 11.2** For the logarithmic utility \(U(w) = \log w, w > 0\), we have \((U')^{-1}(z) = 1/z, z > 0\). Hence

\[ Y = w_0 B(T)/L. \]
(11.5)

**Example 11.3** For the quadratic utility \(U(w) = \beta w - w^2/2\) where \(\beta\) is a constant, we have \((U')^{-1}(z) = \beta - z\). Hence

\[ Y = \beta + \frac{L}{B(T)} \left(\frac{w_0 - \beta E[L/B(T)]}{E[L/B(T)]^2}\right). \]
(11.6)

More concrete formulas can be obtained in a binomial tree model with the parameters \(u, d, r\) and \(p\). The state price density

\[ L = Q/P = \left(\frac{q}{p}\right)^{n_T} \left(\frac{1-q}{1-p}\right)^{T-n_T} \]
is a likelihood ratio, where \( q = (1 + r - d)/(u - d) \), and \( n_t \) is the total number of up moves up to \( t \). The optimal \( Y \) in Example 11.2 is given by

\[
Y = w_0 (1 + r)^T \left( \frac{p}{q} \right)^{n_T} \left( \frac{1 - p}{1 - q} \right)^{T - n_T},
\]

(11.7)

and the optimal expected utility is

\[
E U(Y) = \log w_0 + T \log(1 + r) + pT \log \frac{p}{q} + (1 - p)T \log \frac{1 - p}{1 - q}.
\]

(11.8)

Furthermore, to derive the optimal portfolio \( h \), we again work backwards and set

\[
(1 + r)^T h_0(T) + S(T - 1)u h_1(T) = w_0(1 + r)^T \left( \frac{p}{q} \right)^{n_T-1+1} \left( \frac{1 - p}{1 - q} \right)^{T - n_T - 1},
\]

\[
(1 + r)^T h_0(T) + S(T - 1)d h_1(T) = w_0(1 + r)^T \left( \frac{p}{q} \right)^{n_T-1} \left( \frac{1 - p}{1 - q} \right)^{T - n_T - 1},
\]

which yields

\[
h_1(T) = \frac{w_0(1 + r)^T \left( \frac{p}{q} \right)^{n_T-1} \left( \frac{1 - p}{1 - q} \right)^{T - n_T - 1} (p - q)}{S(T - 1)(u - d) q(1 - q)};
\]

(11.9)

and

\[
h_0(T) = \frac{w_0 \left( \frac{p}{q} \right)^{n_T-1} \left( \frac{1 - p}{1 - q} \right)^{T - n_T - 1 - 1}[u(1 - p)q - d(1 - q)p]}{(u - d) q(1 - q)}.
\]

(11.10)

Since \( V(T - 1) = (1 + r)^{T-1} h_0(T) + S(T - 1)h_1(T) \), it follows that

\[
V(T - 1) = w_0 (1 + r)^{T-1} \left( \frac{p}{q} \right)^{n_T-1} \left( \frac{1 - p}{1 - q} \right)^{T - 1 - n_T - 1},
\]

(11.11)

and the fraction of money invested at time \( T - 1 \) in the stock is given by

\[
\frac{S(T - 1)h_1(T)}{V(T - 1)} = \frac{(1 + r)(p - q)}{(u - d) q(1 - q)}.
\]

(11.12)

Notice that the RHS in (11.12) does not depend on \( n_{T-1} \) and \( T \), also \( V(T - 1) \) in (11.11) has the same form as \( V(T) = Y \) in (11.7). This means that the optimal trading strategy at any time and any state is simply to invest the fraction \( \frac{(1 + r)(p - q)}{(u - d) q(1 - q)} \) of the present wealth in the stock.

Other examples need not enjoy such simplicity but can also be worked out.
Chapter 12

The Binomial Capital Asset Pricing Model

Assume the binomial tree model with parameters $u$, $d$, $r$ and $p$. For each $t = 0, 1, \ldots, T$, denote the conditional expectations $E_P(\cdot | F_t)$ by $E_t(\cdot)$ and $E_Q(\cdot | F_t)$ by $E_{t,Q}(\cdot)$. The corresponding conditional variance and covariance under $P$ are denoted by $\text{var}_t$ and $\text{cov}_t$ respectively. We first consider the mean-variance portfolio problem

(P4) For an arbitrarily fixed $t = 0, 1, \ldots, T - 1$, minimize $\text{var}_t R(t + 1)$ subject to $E_t R(t + 1) = \mu_t$, where $\mu_t$ is a constant satisfying $\mu_t \geq r$ and the portfolio return $R(t + 1)$ is defined by (9.5), or by (9.7),

$$R(t + 1) = \left[ \frac{h_0(t + 1)(1 + r)^t}{V(t)} \right] r + \left[ \frac{h_1(t + 1)S(t)}{V(t)} \right] R_1(t + 1).$$

Note that if $\mu_t = r$, then $\text{var}_t R(t + 1) = 0$ where $R$ is simply the return of the portfolio containing the bank account only. Hence we assume $\mu_t > r$.

The main result in this lecture is

Proposition 12.1 If $R'$ is a solution of (P4) and $R$ is the return of an arbitrary portfolio, then

$$E_t R(t + 1) - r = \beta_t \left[ E_t R'(t + 1) - r \right],$$

where $\beta_t = \frac{\text{cov}_t(R(t+1), R'(t+1))}{\text{var}_t(R(t+1))}$. 

The proof of Proposition 12.1 has several steps.

Step 1. (P4) is equivalent to
(P5) Minimize \( \text{var}_t V(t + 1) \) subject to \( E_t V(t + 1) = w_0(1 + \mu_t) \) and \( V(t) = w_0 \).

**Step 2.** Introducing a Lagrange multiplier \( \beta \), (P5) is equivalent to

(P6) Maximize \( E_t [\beta V(t + 1) - (1/2) V^2(t + 1)] \) subject to \( V(t) = w_0 \).

To verify this, notice that (P6) is the same as minimizing

\[
- E_t [\beta V(t + 1) - (1/2) V^2(t + 1)] \\
= \frac{1}{2} \text{var}_t V(t + 1) + \frac{1}{2} [E_t V(t + 1)]^2 - \beta E_t V(t + 1) \\
= \frac{1}{2} [\text{var}_t V(t + 1) - 2\beta E_t V(t + 1)] + \frac{1}{2} [E_t V(t + 1)]^2,
\]

where the first term matches the objective function for (P5) (with a Lagrange multiplier \( \beta \)), while the second term is equal to \( \frac{1}{2} [w_0(1 + \mu_t)]^2 \) due to the given constraint.

**Step 3.** Define the state price density up to time \( t \) by

\[
L_t = \left( \frac{q}{p} \right)^{n_t} \left( \frac{1-q}{1-p} \right)^{t-n_t},
\]

and the “one-step state price density” between \( t \) and \( t + 1 \) by

\[
l_{t+1} = \frac{L_{t+1}}{L_t} = \left( \frac{q}{p} \right)^{n_{t+1}-n_t} \left( \frac{1-q}{1-p} \right)^{1-(n_{t+1}-n_t)}.
\]

In the current context, \( l_{t+1} \) is a right version of state price density to use.

Following (11.6) in Example 11.3, the optimal solution of (P6) is given by

\[
\hat{V}(t + 1) = \beta - \beta \frac{L_{t+1}}{E_t Q_l t_{t+1}} + w_0(1 + r) \frac{L_{t+1}}{E_t Q_l t_{t+1}},
\]

which implies

\[
E_t \hat{V}(t + 1) = \beta \left[ 1 - (E_t Q_l t_{t+1})^{-1} \right] + w_0(1 + r) (E_t Q_l t_{t+1})^{-1}.
\]

Note that \( E_t Q_l t_{t+1} > 1 \) when \( p \neq q \) (by Jensen’s inequality). Therefore, \( E_t \hat{V}(t + 1) = w_0(1 + \mu_t) \) if and only if

\[
\beta = \frac{w_0[(1 + \mu_t) E_t Q_l t_{t+1} - (1 + r)]}{E_t Q_l t_{t+1} - 1}.
\]

(12.3) specifies a relation between \( \mu_t \) and \( \beta \) such that (P5) and (P6) are equivalent.
If we substitute (12.3) into (12.2) and calculate the corresponding return \( \hat{R}(t + 1) \), we have

\[
\hat{R}(t + 1) = \frac{\mu_t E_{t,Q} l_{t+1} - r}{E_{t,Q} l_{t+1} - 1} - \frac{\mu_t - r}{E_{t,Q} l_{t+1} - 1} l_{t+1},
\]

which means the optimal solution of (P4) is of the form \( a + b l_{t+1} \). Hence Proposition 12.1 follows from Proposition 9.1.

**Note:** Proposition 12.1 is a “look-ahead one step” result, which is essentially the same as a single period model. Nevertheless, the conditioning argument enables us to study the dynamics of a return process. It is also easy to extend the result to more general models than binomial trees, provided the argument based on “conditional independence” still goes through.
Chapter 13

Cash Flows and Forward Prices

We like to broaden our studies in arbitrage pricing from options to other kinds of derivative securities in order to suit the need for quantitative analysis of the evergrowing financial market. Forward contracts, cash flows and their valuation will be considered in this lecture, and futures will be discussed in the next lecture. As you will see, some modifications of the previous concepts and results are necessary although the basic principle still holds.

13.1 Dividends and returns

Assume the basic set-up in Lecture 1. In particular, the bank account \( B \) need not have a constant interest rate \( r(t) \equiv r \). Such a general treatment will be useful later on when considering fixed income securities.

A cash flow can be thought of as a contract which carries a value \( V(t) \) at time \( t \) because the buyer of the contract will receive cash payments from the seller in some future times. A good example of cash flows is the dividends associated with a stock that we mentioned in Section 8.2 without detailed discussion.

For \( n = 1, \ldots, N \) and \( t = 0,1,\ldots,T \), let \( D_n(t) \) be the dividend per unit of security \( n \) issued at time \( t \), in particular \( D_n(0) = 0 \). Let \( S_n(t) \) represent the ex-dividend price of security \( n \), i.e. the price right after any time \( t \) dividend payment. Assume the dividend process is adapted. To check whether arbitrage opportunities exist, it is better to look at returns rather than security prices.

Note that a holder of one unit of security \( n \) at time \( t-1 \) will earn a profit \( \Delta S_n(t) + D_n(t) \) over the period \((t-1,t)\). The definition of return process \( R_n = \{ R_n(t) \} \) given in Section
9.1 is modified as follows: $R_n(0) = 0$, and for $t = 1, \ldots, T$,

$$R_n(t) = \begin{cases} \frac{\Delta S_n(t) + D_n(t)}{S_n(t-1)}, & \text{if } S_n(t-1) > 0 \\ 0, & \text{if } S_n(t-1) = 0. \end{cases}$$ \quad (13.1)$$

Still, $R_0(t) = r(t)$ for the bank account. Moreover, the return process for $S_n^*$ is defined by $R_n^*(0) = 0$, and for $t = 1, \ldots, T$,

$$R_n^*(t) = \begin{cases} \frac{\Delta S_n^*(t) + D_n(t)/B(t)}{S_n^*(t-1)}, & \text{if } S_n^*(t-1) > 0 \\ 0, & \text{if } S_n^*(t-1) = 0. \end{cases}$$ \quad (13.2)$$

Also, $R_0^*(t) \equiv 0$ for the bank account.

If $X = \{X(t), \ t = 0, 1, \ldots, T\}$ is a martingale under a probability measure $Q$ and with respect to $\mathcal{F}$, then the increment process $\Delta X = \{\Delta X(t)\}$ is called the corresponding martingale difference sequence, i.e.

$$E_Q(\Delta X(t) \mid \mathcal{F}_{t-1}) = 0 \ \forall \ t = 1, \ldots, T.$$ 

Here is a modification of Theorem 3.1 in the case of dividend-paying securities.

**Theorem 13.1** No arbitrage $\iff$ there is a probability measure $Q$ with $Q(\omega) > 0 \ \forall \omega \in \Omega$, such that $R_n^* = \{R_n^*(t), \ t = 1, \ldots, T\}$ is a $Q$-martingale difference sequence, $n = 1, \ldots, N$. We still call $Q$ an EMM.

The proof of Theorem 3.1 still works if we replace $\Delta S_n(t)$ there by $\Delta S_n(t) + D_n(t)$.

### 13.2 Forward contracts and prices

A *forward contract* is an agreement made between two parties at time $t$ in which the buyer agrees to buy an underlying asset on a certain specified future date $\tau$ (with $t < \tau \leq T$) for a *delivery price*; while the seller agrees to sell the asset on the same date $\tau$ for the same price. At the maturity date $\tau$, the seller delivers the asset to the buyer in return for a cash payment equal to the delivery price. Although the assets in forward contracts can be commodities, we only focus on securities. The delivery price that would make the time $t$ value of the forward contract to either party zero is called the *forward price*, denoted by $FO(t)$. A basic problem is to determine $FO(t)$ for a forward contract.

Since we only consider those forward contracts that can be replicated by self-financing trading strategies, we assume a complete market without loss of generality. Let $S(t)$ be the
time \( t \) price of an underlying security, and \( b_{t,\tau} = \{ E_Q[B(t)/B(\tau) \mid \mathcal{F}_t]\}^{-1} \) where \( Q \) is an 
EMM. Then we have

\[
FO(t) = S(t) \ b_{t,\tau}.
\] (13.3)

**Note:** The forward price \( FO(t) \) of a forward contract is not the time \( t \) value of the forward contract. The time \( t \) value of the forward contract is zero.

There are a couple of ways to verify (13.3). One is to use risk neutral valuation. Notice that the time \( \tau \) value of the forward contract is \( S(\tau) - FO(t) \). By the definition of \( FO(t) \), we should have 

\[
E_Q\{[S(\tau) - FO(t)] \ B(t)/B(\tau) \mid \mathcal{F}_t\} = 0,
\]

which implies (13.3). The last equality is due to that \( FO(t) \) is measurable with respect to \( \mathcal{F}_t \).

Another way to verify (13.3) is via the standard arbitrage argument. Technically, that would need a zero-coupon bond with maturity \( \tau \). Here we only present the argument in the special case of \( r(t) \equiv r \) thus \( b_{t,\tau} = (1 + r)^{\tau-t} \). This simplification makes the argument more transparent.

Suppose first \( FO(t) < S(t) \ b_{t,\tau} \). An investor can take a long position in the forward contract, short the security at time \( t \), and deposit the proceeds in the bank. At time \( \tau \), the security is purchased under the term of the forward contract for \( FO(t) \), the short position in the security is closed out, and a profit \( S(t) \ b_{t,\tau} - FO(t) > 0 \) is earned.

On the other hand, suppose \( FO(t) > S(t) \ b_{t,\tau} \). Then an investor can borrow the amount \( S(t) \) at time \( t \), buy the security, and take a short position in the forward contract. At time \( \tau \), the security is sold under the term of the forward contract for \( FO(t) \). After using the amount \( S(t) \ b_{t,\tau} \) to repay the loan, a profit \( FO(t) - S(t) \ b_{t,\tau} > 0 \) is realized.

The above result can be generalized to the case of dividend-paying securities. Suppress the subscript \( n \) in the notation given in Section 13.1 and denote the dividend process (a cash flow) by \( D = \{D(t)\} \). At any fixed time \( s > t \), \( D(s) \) is simply a contingent claim, thus its time \( t \) value is \( d_{t,s} = E_Q[D(s)B(t)/B(s) \mid \mathcal{F}_t] \), called the time \( t \) present value of \( D(s) \).

Following Theorem 13.1, we have

\[
E_Q[\Delta S^*(t + 1) + D(t + 1)/B(t + 1) \mid \mathcal{F}_t] = 0 \quad \forall t = 0, 1, \ldots, T - 1; \tag{13.4}
\]

and for \( \tau > t \),

\[
S^*(t) = E_Q\left[ \sum_{s=t+1}^{\tau} \frac{D(s)}{B(s)} + S^*(\tau) \mid \mathcal{F}_t \right]. \tag{13.5}
\]
Hence the time \( t \) present value of a cash flow (such as dividends) \( D(s), t < s \leq \tau \) is given by

\[
\sum_{s=t+1}^{\tau} d_{t,s} = S(t) - E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t].
\] (13.6)

Now we have

**Proposition 13.1**  With a security process \( S \) and a dividend process \( D \), the time \( t \) forward price \( FO(t) \) of a forward contract, which is received and paid at time \( \tau > t \), is given by

\[
FO(t) = \left[ S(t) - \sum_{s=t+1}^{\tau} d_{t,s} \right] b_{t,\tau} = E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] b_{t,\tau}.
\] (13.7)

**Proof**  Once again, we start with the equation

\[
E_Q\{[S(\tau) - FO(t)] B(t)/B(\tau) \mid \mathcal{F}_t\} = 0
\] (13.8)

that defines \( FO(t) \). But the ex-dividend price \( S \) does not satisfy

\[
E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] = S(t).
\]

Instead, it follows from (13.6) that

\[
E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] = S(t) - \sum_{s=t+1}^{\tau} d_{t,s},
\]

which matches \( FO(t) \) after multiplying both sides by \( b_{t,\tau} \).

Alternatively, we may consider the following portfolio that replicates \( S(\tau) \):

At time \( t \): buy one unit of security for \( S(t) \);

Also at time \( t \): for each \( s = t+1, \ldots, \tau \), borrow \( d_{t,s} \) by undertaking the negative of the strategy that replicates the time \( s \) receipt \( D(s) \), then at time \( s \) use the dividend payment \( D(s) \) to settle the liability under the strategy.

The time \( t \) value of this replicating portfolio also amounts to

\[
E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] = S(t) - \sum_{s=t+1}^{\tau} d_{t,s}.
\]

**Example 13.1** Consider the binomial model with parameters \( u, d, r \), and a dividend issued with the constant yield \( \lambda \) on the ex-dividend date \( \tau \), where \( 1 \leq \tau \leq T \).
Suppose $S(\tau - 1) > 0$, then (13.1) yields the return

$$R(\tau) = \frac{\Delta S(\tau) + D(\tau)}{S(\tau - 1)} = u - 1 \text{ or } d - 1,$$

(13.9)

which is the same as in the case without dividends. The same calculation reduces (13.2) to

$$R^*(\tau) = \frac{u}{1 + r} - 1 \text{ or } \frac{d}{1 + r} - 1.$$

(13.10)

According to Theorem 13.1, we can set

$$E_Q[R^*(\tau) | F_{\tau - 1}] = q \left( \frac{u}{1 + r} - 1 \right) + (1 - q) \left( \frac{d}{1 + r} - 1 \right) = 0,$$

(13.11)

and obtain that $q = \frac{1 + r - d}{u - d}$ which is the risk neutral probability provided $d < 1 + r < u$. Therefore, the risk neutral probability $q$ (thus the corresponding EMM $Q$) remains the same as in the case without dividends.

Furthermore, (13.7) gives us the one-step forward price $FO(\tau - 1)$ for a forward contract with maturity $\tau$:

$$FO(\tau - 1) = S(\tau - 1) \cdot (1 + r)(1 - \lambda).$$

(13.12)

Any contingent claim $Y$ defined at $\tau$ has its time $\tau - 1$ value

$$V(\tau - 1) = (1 + r)^{-1}E_Q[Y | F_{\tau - 1}].$$

For example, given $S(\tau - 1) = \zeta$, the time $\tau - 1$ value of an European call option $Y = [S(\tau) - c]^+$ is given by

$$V(\tau - 1) = (1 + r)^{-1}\{q \cdot [u(1 - \lambda)\zeta - c]^+ + (1 - q) \cdot [d(1 - \lambda)\zeta - c]^+]\}.$$  

(13.13)
Chapter 14

Futures Contracts

14.1 Futures vs forward contracts

Like a forward contract, a futures contract is an agreement made at time $t$ which involves a cash payment between two parties in a specified future time with a specified delivery price. However, there are a number of noteworthy differences between futures and forward contracts that we summarize as follows. See Hull’s book for more details.

- **Trading**
  - *Forward*: Traded *over-the-counter* (OTC) anywhere anytime by telephone or other communications between individual buyers and sellers;
  - *Futures*: Traded on some centralized exchange floors, such as Chicago Board of Trade (CBOT) and New York Futures Exchange (NYFE), during their business hours.

- **Standardization**
  - *Forward*: Not standardized — almost everything in a contract is based on private negotiation: the underlying assets to be traded, price, delivery time and location, etc.
  - *Futures*: A standardized contract specifies the underlying asset to be traded, price, delivery date (chosen from among a limited number of dates each year) and location.

- **Delivery**
  - *Forward*: Delivery or final cash settlement usually take place.
  - *Futures*: A very small proportion of contracts are actually delivered. A majority of contracts are closed out prior to maturity.
• Collateral and margin
  
  ○ *Forward:* Collateral level negotiable; no adjustment for daily price fluctuations, settlement only at end of contract. Hence market participants bear the risk of counter-party defaulting.
  
  ○ *Futures:* A central clearinghouse is associated with each exchange to regulate the trade. Each investor is required to initiate a *margin* account as a security deposit. Positions in futures contracts are governed by a daily settlement procedure, referred to as *marking to market.* The investor is allowed to withdraw any balance in the margin account in excess of the initial margin. To ensure that the balance in the margin account never becomes negative, a *maintenance margin,* which is somewhat lower than the initial margin, is set. If the balance in the margin account falls below the maintenance margin, the investor receives a *margin call* and is expected to top up the margin account to the initial margin level the next day. The extra funds deposited are called a *variation margin.* A failure to provide the variation margin would result in closing out the investor’s position, i.e. a broker associated with the clearinghouse would sell the contract.

• Information and *liquidity* (important to statistical analysis of financial market, sometimes referred to as *empirical finance*)
  
  ○ *Forward:* No volume information available; low liquidity due to a large variety of contract terms.
  
  ○ *Futures:* Volume information published; high liquidity due to standardized contracts.

### 14.2 Futures prices

Marking to market is a device in futures market to reduce the risk of default inherent in forward contracts. Recall that the value of a forward contract need not equal zero except at the time when the contract starts. In contrast, the value of a futures contract is maintained at zero at all times via marking to market. Thus either party can close out his/her position at any time.

The futures price of a futures contract, specified at the time when the futures contract is entered into, is not like the forward price which is defined by the equation (13.8). Instead, it is determined on a given futures exchange by the usual law of supply and demand, similar to stock prices.

For $m = 1, \ldots, M$, let $FU_m(t, t_m)$ be the futures price of a futures contract with starting date $t$ and maturity date $t_m$, and $\Delta FU_m(s, t_m) = FU_m(s, t_m) - FU_m(s - 1, t_m)$, $t < s \leq t_m$. 

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An investor taking a long position either receives a payment \( \Delta F U_m(s, t_m) \) at time \( s \), or makes a payment \(-\Delta F U_m(s, t_m)\) if the futures price falls. These fluctuations are reflected in marking to market. We will study the connections between

(i) the futures price process \( F U_m(\cdot, t_m) \) and the underlying security process \( S_m(\cdot) \);

(ii) the futures price \( F U_m(t, t_m) \) and the corresponding forward price \( F O(t) \) with the same maturity \( t_m \).

Let \((h; h^f) = (h_0, h_1, \ldots, h_N; h^f_1, \ldots, h^f_M)\) denote the portfolio process in a combined spot/futures market, where all the components are predictable processes. For each \( m \), \( h^f_m(t) \) represents the position in futures contract \( m \) held from time \( t - 1 \) to \( t \). We assume that \( F U_m(t, t_m) = 0 \) and \( h^f_m(t) = 0 \) for any \( t > t_m \).

Consider the value process. The time \( t \) value of a portfolio right before marking to market is

\[
V(t) = h_0(t)B(t) + \sum_{n=1}^{N} h_n(t)S_n(t) + \sum_{m=1}^{M} h^f_m(t)\Delta F U_m(t, t_m); \quad (14.1)
\]

and the time \( t \) value of the portfolio right after marking to market is

\[
V^+(t) = h_0(t+1)B(t) + \sum_{n=1}^{N} h_n(t+1)S_n(t). \quad (14.2)
\]

Compared to (14.1), the absence of terms involving futures prices in (14.2) is due to the updating adjustment via marking to market. A trading strategy \((h; h^f)\) is said to be self-financing if its value process satisfies

\[
V(t) = V^+(t) \quad \forall \ t = 1, \ldots, T - 1. \quad (14.3)
\]

Since a futures trader must have positive wealth, we cannot define arbitrage by starting with \( V(0) = 0 \). Nevertheless, other definitions of arbitrage would work. An arbitrage opportunity in a spot/futures market corresponds to a self-financing strategy \((h; h^f)\) whose value satisfies

(a) \( V^*(T) \geq V^*(0) \) \( \forall \omega \), and

(b) \( V^*(T) > V^*(0) \) for some \( \omega \).

An alternative to (a) and (b) is

(a') \( G^*(T) \geq 0 \) \( \forall \omega \), and

(b') \( G^*(T) > 0 \) for some \( \omega \),

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where the discounted gain $G^*$ is specified through its increment in $(t - 1, t]$

$$
\Delta G^*(t) = \sum_{n=1}^{N} h_n(t) \Delta S_n^*(t) + \sum_{m=1}^{M} h_m^f(t) \Delta F(t) U_m(t, t_m)/B(t), \quad t = 1, \ldots, T. \quad (14.4)
$$

The following theorem is an extension of Theorem 3.1 to the case of spot/futures market.

**Theorem 14.1** For a combined spot/futures market, no arbitrage $\iff$ there is an EMM $Q$, such that for every $n = 1, \ldots, N$, the process $S_n^*$ is a $Q$-martingale; moreover, for every $m = 1, \ldots, M$,

$$
E_Q[\Delta F(t) U_m(t, t_m)/B(t) | \mathcal{F}_{t-1}] = 0, \quad t = 1, \ldots, T. \quad (14.5)
$$

The proof is left as an exercise, since the argument in proving Theorem 3.1 can be modified straightforwardly.

**Corollary 14.1** Consider the following special cases of Theorem 14.1.

(a) Suppose $B$ is a predictable process. Then (14.5) is equivalent to that $F_U(t, t_m)$ is a $Q$-martingale for each $m$.

(b) Suppose $B$ is deterministic with $r(t) \equiv r$. Then for every $m$ and $t$,

$$
F_U(t, t_m) = FO(t), \quad (14.6)
$$

where $FO(t)$ is the forward price of a forward contract starting at $t$ and ending at $t_m$.

**Proof** (a) is trivial. For (b), note that $F_U(t_m, t_m) = S_m(t_m)$, i.e. the futures price converges to the spot price of the underlying security at the delivery date. The martingale properties of $F_U(t, t_m)$ and $S_m(t)$ imply that

$$
F_U(t, t_m) = E_Q[F_U(t_m, t_m) | \mathcal{F}_t] = E_Q[S_m(t_m) | \mathcal{F}_t]
= (1 + r)^{t_m-t} S_m(t) = FO(t).
$$

**Note:** (14.6) remains valid when underlying securities pay dividends. This just follows from (13.7).

**Example 14.1** For a single period market, (14.6) always holds even when $B$ is random, adaptive and nonpredictable. This is intuitively obvious since there is no marking to market. The holder of a futures contract can only wait until the delivery date — same as in a similar forward contract. To verify (14.6), note that (14.5) in this case is written as

$$
E_Q[\Delta F(t) U(1, 1)/B(1) | \mathcal{F}_0] = E_Q[[S_m(1) - F_U(0, 1)]/B(1) | \mathcal{F}_0] = 0,
$$
which is just the equation to define the forward price $FO(0)$.

**Example 14.2** Consider a binomial tree model with $T = 2$. Identify $\omega_1 = uu$, $\omega_2 = ud$, $\omega_3 = du$ and $\omega_4 = dd$. Let $S(0) = 1$, $u = 1.07$, $d = 0.92$; $B(0) = B(1) = 1$ and $B(2) = \begin{cases} 1.03, & \text{on } \{\omega_1, \omega_2\} \\ 1.04, & \text{on } \{\omega_3, \omega_4\} \end{cases}$

Since there is only a single stock, we suppress the subscript $m$, and calculate $Q$, $FU(0, 2)$ and $FO(0)$ (also with delivery date $T = 2$).

- To calculate $Q$, write $E_Q[S(t)/B(t) \mid \mathcal{F}_{t-1}] = S(t-1)/B(t-1)$:
  - For $t = 2$:
    \begin{align*}
    1.07q_1 + 0.92(1 - q_1) &= 1.03 \text{ thus } q_1 = \frac{11}{15} \text{ on } \{\omega_1, \omega_2\}; \\
    1.07q_2 + 0.92(1 - q_2) &= 1.04 \text{ thus } q_2 = \frac{12}{15} \text{ on } \{\omega_3, \omega_4\}.
    \end{align*}
  - For $t = 1$:
    \begin{align*}
    q_0 &= \frac{1-d}{u-d} = \frac{8}{15}.
    \end{align*}
  - Hence $Q(\omega_i), i = 1, 2, 3, 4$, are equal to $q_0q_1$, $q_0(1 - q_1)$, $(1 - q_0)q_2$ and $(1 - q_0)(1 - q_2)$ respectively.

- (14.5) is carried out in two steps.
  - For $t = 2$, $S(1)/B(1) = FU(1, 2) E_Q[B^{-1}(2) \mid \mathcal{F}_1]$, i.e.
    \begin{align*}
    FU(1, 2) &= 1.07 \cdot 1.03 = 1.0721 \text{ on } \{\omega_1, \omega_2\}; \\
    FU(1, 2) &= 0.92 \cdot 1.04 = 0.9568 \text{ on } \{\omega_3, \omega_4\}.
    \end{align*}
  - For $t = 1$,
    \begin{align*}
    FU(0, 2) &= E_QFU(1, 2) = 1.1021q_0 + 0.9568(1 - q_0) = \frac{15544}{150000} \approx 1.03429.
    \end{align*}

- (13.3) yields
  \[
  FO(0) = S(0) \left[ E_Q B^{-1}(2) \right]^{-1} = \left[ 1.03^{-1} q_0 + 1.04^{-1}(1 - q_0) \right]^{-1} = \frac{169680}{155300} \approx 1.03464.
  \]

In this example, $B$ is random and predictable, but $FU(0, 2) \neq FO(0)$.

### 14.3 Options on futures

Options or more general contingent claims can be defined as derivatives of underlying securities and/or futures in a combined spot/futures market. Risk neutral valuation based on Theorem 14.1 can be carried out. Any marketable contingent claim can be hedged via
replicating portfolios \((h; h^f)\). The only technical subtlety to keep in mind is that when using an EMM \(Q\), the discounted security price \(S^*\) is a \(Q\)-martingale while the undiscounted futures price \(FU(\cdot, \tau)\) is a \(Q\)-martingale (assuming \(B\) is predictable). Hence the price of a contingent claim depends on whether it is defined as a derivative of a security or a futures contract. We will give some related problems in Homework 3.
Chapter 15

Zero-coupon Bonds, Yields and Forward Rates

The fixed-income market is an important part of the global financial market in which various interest rate securities, such as bonds, are traded. The total volume of fixed-income securities traded in the market is much greater than that of equities such as common stocks. Mathematical models for fixed-income derivatives are also more subtle and complex. We will introduce the basic concepts and some useful models in the next few lectures.

Assume the basic setting in Section 1.1: a sample space $\Omega$, an underlying probability measure $P$, a filtration $\mathcal{F}$ with finite horizon $T$. In particular, the bank account $B$ is assumed to be random and predictable with interest rate $r(t) > 0$ for all $t = 1, \ldots, T$. This means that the riskless interest rate $r(t)$ for borrowing or lending over the period $(t - 1, t]$ is known at time $t - 1$. The process $B$ is taken as a numeraire. $r(t)$ is called the spot rate or short rate.

- Various bonds are considered as risky securities. One of them is a collection of zero-coupon or discount bonds, denoted by $\{B(t, \tau) : 0 \leq t \leq \tau, \tau = 1, \ldots, T\}$, where $B(t, \tau)$ represents the time $t$ price of a zero-coupon bond with maturity $\tau$. Sometimes we simply call a zero coupon bond with maturity $\tau$ a $\tau$-bond. Assume that for each $\tau$, the process $B(\cdot, \tau)$ is positive and adapted to $\mathcal{F}$, in particular, $B(\tau, \tau) = 1$ at par (the nominal value is $1$ at maturity). On the other hand, for each $t$, the collection $\{B(t, \tau), t < \tau \leq T\}$ is called the time $t$ term structure of zero-coupon bond prices. Hence we are dealing with a process $B(\cdot, \cdot)$ with double indices, which makes the analysis considerably more challenging.

Now we consider a couple of other term structures equivalent to $B(t, \cdot)$, and we assume $t \leq \tau$.

- Let $Y(t, \tau)$ be the constant interest rate (or return) at which $B(t, \tau)$, when compounded
during \((t, \tau]\), would reach $1 at time \(\tau\), called the \textit{yield to maturity}, i.e.

\[
B(t, \tau) \left[ 1 + Y(t, \tau) \right]^{\tau - t} = 1,
\]

or equivalently,

\[
Y(t, \tau) = \left[ B(t, \tau) \right]^{-1} - 1.
\]

In particular, \(Y(t - 1, t) = r(t)\), the spot rate at one period before maturity. [Why? See (15.7).] For each \(t\), the collection \(\{Y(t, \tau), t < \tau \leq T\}\) is called the \textit{time \(t\) term structure of interest rates or yield curve}. The two term structures \(B(t, \cdot)\) and \(Y(t, \cdot)\) are equivalent.

- Let \(f(t, \tau)\) be the “short rate” such that
  
(i) it is locked into at time \(t\);

(ii) it is applied to the period \((\tau, \tau + 1]\);

(iii) it is associated with a \((\tau + 1)\)-bond.

Then we have

\[
\frac{B(t, \tau + 1)}{B(t, \tau)} \left[ 1 + f(t, \tau) \right] = 1,
\]

or equivalently,

\[
f(t, \tau) = \frac{B(t, \tau)}{B(t, \tau + 1)} - 1.
\]

To see why this is true, assume there exists an EMM \(Q\) such that for every \(\tau\), the discounted \(\tau\)-bond process \(B^\tau(\cdot, \tau)\) is a \(Q\)-martingale, i.e.

\[
B(t - 1, \tau) = E_Q[B(t, \tau)B(t - 1)/B(t) \mid \mathcal{F}_{t-1}], \quad 1 \leq t \leq \tau.
\]

Notice that in a fixed income market, zero coupon bonds play the same role as risky securities like common stocks in a stock market. We can consider a forward contract on a \((\tau + 1)\)-bond, delivered at \(\tau\). According to (13.3), its forward price \(FO(t)\) at time \(t\) is given by

\[
FO(t) = B(t, \tau + 1) b_{t, \tau} = \frac{B(t, \tau + 1)}{E_Q[B(t)/B(\tau) \mid \mathcal{F}_t]} = \frac{B(t, \tau + 1)}{B(t, \tau)},
\]

where the last equality is due to that

\[
B(t, \tau) = E_Q[B(t)/B(\tau) \mid \mathcal{F}_t].
\]

In particular,

\[
f(t, t) = r(t + 1).
\]
Exercise 15.1 Verify the following results:

\[ B(t, \tau) = E_Q\{ [(1 + r(t + 1)) \cdots (1 + r(\tau))]^{-1} | \mathcal{F}_t \}; \quad (15.9) \]

\[ r(t) = B^{-1}(t - 1, t) - 1; \quad (15.10) \]

\[ B(t, \tau) = \prod_{s=t+1}^{\tau} [1 + f(t, s - 1)]^{-1}; \quad (15.11) \]

For every \( t \), \( B(t, \tau) \) is strictly decreasing in \( \tau \).
Chapter 16

Examples of Term Structure Models

There are two different kinds of term structure models: spot rate models (short rate models, equilibrium models, etc.), in which initial term structures are outputs from the models; and yield curve models (forward rate models, no arbitrage models, etc.), in which initial term structures are inputs to the models. Before general studies of these two approaches, we illustrate them in two simple examples respectively in this lecture.

In both examples, we assume $T = 3$, $\Omega = \{\omega_i, \ i = 1, 2, 3, 4\}$, and the following partitions that generate the filtration $\mathcal{F}$:

$\mathcal{P}_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\}$;

$\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$;

and

$\mathcal{P}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$.

Example 16.1

• Start with a spot rate binomial tree, Figure 16.1, in which

$r(1) = 0.04$;

$r(2) = \begin{cases} 0.06 & \text{on } \{\omega_1, \omega_2\}, \\ 0.03 & \text{on } \{\omega_3, \omega_4\}; \end{cases}$
Figure 16.1: Spot rate tree
and

\[ r(3) = \begin{cases} 
0.07 & \text{on } \{ \omega_1 \}, \\
0.05 & \text{on } \{ \omega_2 \}, \\
0.04 & \text{on } \{ \omega_3 \}, \\
0.02 & \text{on } \{ \omega_4 \}. 
\end{cases} \]
• By (15.10) \( B(t - 1, t) = [1 + r(t)]^{-1} \), we obtain that

\[
B(0, 1) = 0.9615;
\]

\[
B(1, 2) = \begin{cases} 
0.9434 & \text{on } \{\omega_1, \omega_2\}, \\
0.9709 & \text{on } \{\omega_3, \omega_4\}
\end{cases}
\]

and

\[
B(2, 3) = \begin{cases} 
0.9346 & \text{on } \{\omega_1\}, \\
0.9524 & \text{on } \{\omega_2\}, \\
0.9615 & \text{on } \{\omega_3\}, \\
0.9804 & \text{on } \{\omega_4\}
\end{cases}
\]

• At this point, we have some flexibility. We choose an EMM \( Q \) by assigning \( Q(\omega_i) = 1/4, \ i = 1, 2, 3, 4 \).

• It follows from (15.7) or (15.9) that

\[
B(0, 2) = E_Q[1/B(2)] = \frac{1}{1.04} \cdot \frac{1}{2} \cdot \left( \frac{1}{1.06} + \frac{1}{1.03} \right) = 0.9203,
\]

\[
B(0, 3) = E_Q[1/B(3)] = \frac{1}{1.04} \cdot \frac{1}{4} \cdot \left( \frac{1}{1.06 \cdot 1.07} + \frac{1}{1.06 \cdot 1.05} + \frac{1}{1.03 \cdot 1.04} + \frac{1}{1.03 \cdot 1.02} \right) = 0.8811,
\]

and

\[
B(1, 3) = E_Q \left\{ \frac{1}{[1 + r(2)][1 + r(3)]} \mid \mathcal{F}_1 \right\} = E_Q \left\{ \frac{1}{1 + r(2)} \mid \mathcal{F}_1 \right\} = \left\{ \frac{1}{1.06 \cdot \frac{1}{2}} \left( \frac{1}{1.07} + \frac{1}{1.05} \right) = 0.8901 \right\} \text{ on } \{\omega_1, \omega_2\},
\]

\[
\left\{ \frac{1}{1.03 \cdot \frac{1}{2}} \left( \frac{1}{1.07} + \frac{1}{1.05} \right) = 0.9427 \right\} \text{ on } \{\omega_3, \omega_4\}.
\]

The result is summarized in Table 16.1.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( B(0, 1) )</th>
<th>( B(0, 2) )</th>
<th>( B(1, 2) )</th>
<th>( B(0, 3) )</th>
<th>( B(1, 3) )</th>
<th>( B(2, 3) )</th>
<th>( Q )</th>
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<tr>
<td>( \omega_1 )</td>
<td>0.9615</td>
<td>0.9203</td>
<td>0.9434</td>
<td>0.8811</td>
<td>0.8901</td>
<td>0.9346</td>
<td>0.25</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>~</td>
<td>~</td>
<td>0.9709</td>
<td>~</td>
<td>0.9427</td>
<td>0.9615</td>
<td>~</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>0.9804</td>
<td>~</td>
</tr>
</tbody>
</table>

Table 16.1: Data for Example 16.1. The notation ~ represents the same number as above.
A drawback of the approach used in Example 16.1 is that the initial (time 0) term structure is derived from the model, thus need not be consistent with the given data (the initial term structure data is available at time 0). An alternative approach is to model the entire family of yield curves *conditioning* on the given initial term structure.

**Example 16.2**

- Suppose the initial term structure is given by
  
  \[ B(0, 1) = 0.96 \quad \text{[thus \ } r(1) = 0.0417], \]
  
  \[ B(0, 2) = 0.92, \]
  
  and
  
  \[ B(0, 3) = 0.88. \]

- If we let \( Q(\omega_i) = 1/4, \ i = 1, 2, 3, 4 \) as in Example 16.1, there is still some flexibility in specification of \( r(2) \) and \( r(3) \), or equivalently, \( B(t, \cdot), t = 1, 2 \). We will move forward one period at a time.

- For \( t = 2 \), set \( r(2) = 0.06 \) on \( \{\omega_1, \omega_2\} \). Then
  
  \[ B(0, 2) = \frac{1}{1 + r(1)} \ E_Q \left[ \frac{1}{1 + r(2)} \right], \]
  
  i.e.
  
  \[ 0.92 = \frac{1}{1.0417} \cdot \frac{1}{2} \left[ \frac{1}{1.06} + \frac{1}{1 + r(2)} \right], \]
  
  which gives rise to
  
  \[ r(2) = 0.0274 \quad \text{on} \ \{\omega_3, \omega_4\}. \]

  Hence
  
  \[ B(1, 2) = \frac{1}{1 + r(2)} = \begin{cases} 0.9434 & \text{on} \ \{\omega_1, \omega_2\}, \\ 0.9733 & \text{on} \ \{\omega_3, \omega_4\}. \end{cases} \]

- For \( t = 3 \), set

  \[ r(3) = \begin{cases} 0.07 & \text{on} \ \{\omega_1\}, \\ 0.05 & \text{on} \ \{\omega_2\}, \\ 0.04 & \text{on} \ \{\omega_3\}. \end{cases} \]
Notice the constraint

\[ B(0,3) = \frac{1}{1 + r(1)} \ E_Q \left\{ \frac{1}{1 + r(2)} \left[ 1 + r(3) \right] \right\}, \]

i.e.

\[ 0.88 = \frac{1}{1.0417} \cdot \frac{1}{4} \left[ \frac{1}{1.06 \cdot 1.07} + \frac{1}{1.06 \cdot 1.05} + \frac{1}{1.0274 \cdot 1.04} + \frac{1}{1.0274} \cdot \frac{1}{1 + r(3)} \right], \]

which implies

\[ r(3) = 0.0238 \] \text{ on } \{\omega_4\}.

Therefore,

\[ B(2,3) = \frac{1}{1 + r(3)} = \begin{cases} 0.9346 & \text{on } \{\omega_1\}, \\ 0.9524 & \text{on } \{\omega_2\}, \\ 0.9615 & \text{on } \{\omega_3\}, \\ 0.9768 & \text{on } \{\omega_3\}. \end{cases} \]

Finally,

\[ B(1,3) = \frac{1}{1 + r(2)} \ E_Q \left[ \frac{1}{1 + r(3)} \right] \mathcal{F}_1 \]

\[ = \left\{ \begin{array}{c} \frac{1}{1.06} \cdot \frac{1}{2} \left( \frac{1}{1.07} + \frac{1}{1.05} \right) = 0.8901 \quad \text{on } \{\omega_1,\omega_2\}, \\ \frac{1}{1.0274} \cdot \frac{1}{2} \left( \frac{1}{1.04} + \frac{1}{1.0238} \right) = 0.9433 \quad \text{on } \{\omega_3,\omega_4\}. \end{array} \right. \]

The result is summarized in Table 16.2.

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(B(1,2))</th>
<th>(B(1,3))</th>
<th>(B(2,3))</th>
<th>(r(2))</th>
<th>(r(3))</th>
<th>(Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_1)</td>
<td>0.9434</td>
<td>0.8901</td>
<td>0.9346</td>
<td>0.06</td>
<td>0.07</td>
<td>0.25</td>
</tr>
<tr>
<td>(\omega_2)</td>
<td>\sim</td>
<td>\sim</td>
<td>0.9524</td>
<td>\sim</td>
<td>0.05</td>
<td>\sim</td>
</tr>
<tr>
<td>(\omega_3)</td>
<td>0.9733</td>
<td>0.9433</td>
<td>0.9615</td>
<td>0.0274</td>
<td>0.04</td>
<td>\sim</td>
</tr>
<tr>
<td>(\omega_4)</td>
<td>\sim</td>
<td>\sim</td>
<td>0.9768</td>
<td>\sim</td>
<td>0.0238</td>
<td>\sim</td>
</tr>
</tbody>
</table>

Table 16.2: Data for Example 16.2. The notation \(\sim\) represents the same number as above.

**Notes:**

(a) In Example 16.2, one can first specify all \(B(\cdot,\cdot)\) in a consistent manner without using \(Q\) and \(r(\cdot)\). Then \(Q\) and \(r(\cdot)\) will be determined accordingly.

(b) In practice, *trinomial* trees or bushy trees with even more branches from each node are often used to study *multi-factor* term structure models (later).
Chapter 17

Spot Rate Modelling via Markov Chains and Stochastic Difference Equations

We introduce two methods in modelling spot rates: Markov chains (MC) and stochastic difference equations (SDE). Each method has its own advantage. MC makes use of binomial trees, and SDE is naturally linked to models in continuous-time finance. Overall, MC provides somewhat more general models than SDE.

17.1 Spot rate Markov chains

A stochastic process $X = \{X(t), \ t = 0,1,\ldots,T\}$ is called a Markov chain with a state space $S$ if for every $t = 0,1,\ldots,T-1$,
\[
P\{X(t+1) = x_{t+1} \mid X(s) = x_s, \ 0 \leq s \leq t\} = P\{X(t+1) = x_{t+1} \mid X(t) = x_t\}
\]
for all possible values $x_s \in S$, $0 \leq s \leq t + 1$. Several quantities of interest, such as the spot rate $r(\cdot)$ and the zero-coupon bond $B(\cdot,\cdot)$ can be modelled as Markov chains. A convenient way is to construct an underlying Markov chain $X$ such that $r(\cdot)$, $B(\cdot,\cdot)$, etc. are defined as appropriate functions of $X$. For example, we let $r(t+1) = g_t(n_t)$ with a strictly positive and increasing function $g_t$ for each $t$.

Let $S = \{(t,n_t): n_t = 0,1,\ldots,t; \ t = 0,1,\ldots,T\}$, where $n_t$ denotes the number of up moves up to $t$ in the binomial tree in Figure 17.1. The precise meaning of $n_t$ is not crucial. It can be thought of as coding a number of exogenous factors that affects $r(\cdot)$, $B(\cdot,\cdot)$, etc. in a fixed income market. Assume that $X$ is a Markov chain under an EMM $Q$, with transition
Figure 17.1: Binomial tree of Markov chain $X$ and spot rate $r$
probabilities

\[ q(t, n_t) = Q\{ X(t + 1) = (t + 1, n_t + 1) \mid X(t) = (t, n_t) \} = 1 - Q\{ X(t + 1) = (t + 1, n_t) \mid X(t) = (t, n_t) \}, \]

i.e. \( n_{t+1} = n_t + 1 \) or \( n_t \) are the only allowable transitions. Since \( r(\cdot) \) is predictable, the value of spot rate \( r(t + 1) \) applied to the period \( (t, t + 1] \) is known at time \( t \), denoted by \( r(t + 1, n_t) \).

Note that this formulation forces a recombining tree as shown in Figure 17.1, but this binomial tree is inhomogeneous in the sense that the transition probabilities \( q \) take different values at different nodes.

A full specification of the model involves \( T(T + 1) \) parameters: one value of \( q \) and one value of \( r \) at each node, with total number of \( T(T + 1)/2 \) nodes (not counting the terminal nodes at \( T \)). The following special case illustrates this.

**Example 17.1** Assume \( q(t, n_t) \equiv \frac{1}{2} \forall (t, n_t) \in S \). Moreover, the initial term structure \( \{ B(0, \tau), \tau = 1, \ldots, T \} \) adds \( T \) constraints. There are still \( T(T - 1)/2 \) parameters to be specified [among \( T(T + 1)/2 \) values of \( r \)]. It is often useful to reparametrize the model by specifying the spot rate volatilities

\[ \sigma(t, n_t) = \sqrt{q(t, n_t) [1 - q(t, n_t)]} \log \frac{r(t + 2, n_t + 1)}{r(t + 2, n_t)} = \frac{1}{2} \log \frac{r(t + 2, n_t + 1)}{r(t + 2, n_t)} \quad (17.1) \]

for \( t = 0, 1, \ldots, T - 2 \). [Is \( \sigma(t, n_t) > 0 \forall (t, n_t) \) ? Why?]

If we let \( B(t, n_t; \tau) \) denote the value of \( B(t, \tau) \) at the state \((t, n_t)\), then (15.5) implies that

\[ B(t, n_t; \tau) = \frac{q(t, n_t) B(t + 1, n_t + 1; \tau) + [1 - q(t, n_t)] B(t + 1, n_t; \tau)}{1 + r(t + 1, n_t)}. \quad (17.2) \]

### 17.2 Stochastic difference equations

Let the spot rate \( r \) be governed by the SDE

\[ \Delta r(t + 1) = \mu(t, r(t)) + \sigma(t, r(t)) \epsilon_t, \quad (17.3) \]

where \( \mu \) is a real-valued function and \( \sigma \) a positive function; \( \Delta r(t + 1) = r(t + 1) - r(t) \), and \( \epsilon_t, \ t = 0, 1, \ldots, T - 1 \) are iid random variables with \( Q(\epsilon_t = 1) = Q(\epsilon_t = -1) = \frac{1}{2} \). Obviously, under \( Q \) the increment \( \Delta r(t + 1) \) has the conditional mean \( \mu(t, r(t)) \) and the conditional variance \( \sigma^2(t, r(t)) \) given \( r(t) \).
To see the connections between SDE and MC, first consider a special case of MC in which 
$q(t, n_t) \equiv \frac{1}{2} \forall (t, n_t) \in S$. Then the increments $n_t - n_{t-1}$ are iid Bernoulli random variables under $Q$ thus we can set 

$$
\epsilon_t = 2(n_t - n_{t-1}) - 1. \tag{17.4}
$$

The SDE (17.3) is expressed as 

$$
\begin{align*}
    r(t+1, n_t + 1) - r(t, n_t - 1) &= \mu(t, r(t, n_t - 1)) + \sigma(t, r(t, n_t - 1)) \\
    r(t+1, n_t - 1) - r(t, n_t + 1) &= \mu(t, r(t, n_t - 1)) - \sigma(t, r(t, n_t - 1)),
\end{align*}
$$

in which case $\mu$ and $\sigma$ should satisfy 

$$
\begin{align*}
    \mu(t, r(t, n_t - 1)) &= \frac{1}{2} [r(t+1, n_t - 1) + r(t+1, n_t - 1) - 2r(t, n_t - 1)] \\
    \sigma(t, r(t, n_t - 1)) &= \frac{1}{2} [r(t+1, n_t - 1) - r(t+1, n_t - 1)],
\end{align*}
$$

so that the spot rate $r$ is governed by (17.3).

On the other hand, if $r$ satisfies the SDE (17.3), then clearly $r$ is a Markov chain. However, there are a number of more interesting and subtle questions to be answered:

**Q1** Under what conditions on $\mu$ and $\sigma$, an underlying MC $X$ can be represented by a recombining binomial tree as in Figure 17.1 such that $r$ is defined by $r(t+1) = g_t(n_t)$?

**Q2** What functions $\mu$ and $\sigma$ will assure a positive spot rate $r$ defined via (17.3)?

**Q3** What functions $\mu$ and $\sigma$ will guarantee that the spot rate $r$ defined via (17.3) satisfies the _mean reversion_ property, i.e. $r$ tends to decrease when it is above a threshold and increase when it is below the threshold?

We will discuss these issues in the next couple of lectures when we study some well-known special models.
Chapter 18

Vasicek, Cox-Ingersoll-Ross, and Hull-White Models

Different expressions of the drift $\mu$ and volatility $\sigma$ in the SDE (17.3)

$$\Delta r(t + 1) = \mu(t, r(t)) + \sigma(t, r(t)) \; \epsilon_t$$

lead to various spot rate models. In this lecture, we first consider time-homogeneous SDEs in which $\mu(t, r(t)) = \mu(r(t))$ and $\sigma(t, r(t)) = \sigma(r(t))$, i.e. the drift and volatility do not involve time $t$ explicitly. We then study an inhomogeneous SDE in Hull-White model, along with Markov chains represented by binomial or trinomial trees.

18.1 Vasicek and CIR models

Consider the following special cases in which $a, b, \sigma$ are all positive constants, and $\beta$ is a constant with $0 \leq \beta \leq 1$.

**Example 18.1 (Vasicek model)** $\mu(t, r(t)) = a[b - r(t)], \; \sigma(t, r(t)) = \sigma.$

**Example 18.2 [Cox-Ingersoll-Ross (CIR) model]** $\mu(t, r(t)) = a[b - r(t)], \; \sigma(t, r(t)) = \sigma \sqrt{r(t)}.$

**Example 18.3 (a more general class)** $\mu(t, r(t)) = a[b - r(t)], \; \sigma(t, r(t)) = \sigma \; [r(t)]^\beta.$

A common feature of these models is the mean reversion property: the spot rate $r$ tends to decrease if $r > b$ and increase if $r < b$. An advantage of CIR over Vasicek is its capability of enforcing positive values on $r$. To formulate the question more precisely, assume $r(1) = x > 0$ and let

$$\tau_x = \inf \{2 \leq t \leq T : r(t) \leq 0\}. \quad (18.1)$$
The question is: under what conditions on the parameters $a, b$ and $\sigma$ in the SDEs for Vasicek and CIR respectively, $Q(\tau_x \leq T) = 0$ will be satisfied? Note that $Q(\tau_x \leq T) = 0$ and $P(\tau_x \leq T) = 0$ are equivalent since $Q$ is an EMM. Both imply that the values of $r$ remain positive in the whole process. It turns out that those conditions needed in CIR are quite mild and reasonable, but the conditions needed in Vasicek are too strict to be realistic. More on this later.

18.2 Hull-White model

In a very general form, Hull-White model can be expressed via the SDE

$$\Delta r(t + 1) = a(t) [b(t) - r(t)] + \sigma(t) [r(t)]^\beta \epsilon_t,$$

(18.2)

where $\beta \geq 0$ is still a constant, but $a(t), b(t)$ and $\sigma(t)$ are positive-valued deterministic functions. These functions greatly enhance modelling flexibility, e.g. the initial term structure can be incorporated into $a(t)$ and $b(t)$. The condition on $\{\epsilon_t\}$ is often relaxed in this model, e.g. $\epsilon_t$’s are independent with $Q(\epsilon_t = 1) = q(t) < 1$.

Under what conditions on functions $a, b$ and $\sigma$, an underlying MC $X$ can be represented by a recombining tree such that $r$ is defined by $r(t + 1) = g_t(n_t)$ (Q1 in Lecture 17)? A recombining tree requires that starting from any node, a “up-down” combination ($\epsilon_t = 1$ and $\epsilon_{t+1} = -1$) and a “down-up” combination ($\epsilon_t = -1$ and $\epsilon_{t+1} = 1$) should merge at the same node. In other words, starting from $r(t)$, these two combinations must arrive at the same value of $r(t + 2)$. Standard calculation turns this into the condition

$$\sigma(t + 1) = [1 - a(t + 1)] \sigma(t) \quad \forall t,$$

(18.3)

which also requires $a(t) < 1 \forall t$ (for positive volatility).

Trinomial trees are often used in this situation to provide an extra degree of freedom. In a trinomial tree, three branches come out of each node: up, middle and down. There are $2t + 1$ nodes at time $t$. The recombining assumption consists of “up-down = middle-middle = down-up”, “up-middle = middle-up” and “down-middle = middle-down”. The three possible moves up, middle and down correspond to $\epsilon_t = 1$, $\epsilon_t = 0$ and $\epsilon_t = -1$, with risk neutral probabilities $Q(\epsilon_t = 1) = Q(\epsilon_t = -1) = q(t) < 1/2$ and $Q(\epsilon_t = 0) = 1 - 2q(t)$. The same argument shows that the recombining condition is still given by (18.3).

18.3 Refined lattice and SDE

So far in Lectures 17 and 18, the binomial (or trinomial tree) models and SDEs for spot rates have fixed integer 1 as the span for both time increment and space increment. Refinement of
this is needed for more realistic models, especially for passing the limit to continuous-time finance. To carry out this technical step, fix $T > 0$ and positive integer $n$, let $\delta = T/n$ be the time increment and $\sqrt{\delta}$ the space increment in the binomial or trinomial trees. Accordingly, for $j = 0, 1, \ldots, n - 1$, we let

$$r_\delta(j) = r(j\delta), \quad \Delta r_\delta(j + 1) = r_\delta(j + 1) - r_\delta(j).$$

The functions $\mu_\delta(j, r_\delta(j))$, $\sigma_\delta(j, r_\delta(j))$, $a_\delta(j)$ and $b_\delta(j)$ are defined in the same way. The SDE (17.3) becomes

$$\Delta r_\delta(j + 1) = \mu_\delta(j, r_\delta(j)) \delta + \sigma_\delta(j, r_\delta(j)) \sqrt{\delta} \epsilon_j. \quad (18.4)$$

Keep in mind that for a fixed $\delta$ (or $n$), the basic setting in Lecture 1 is still valid. However, if we let $n \to \infty$ (thus $\delta \to 0$), many complicated technical issues will need to be taken care of.
Chapter 19

Heath-Jarrow-Morton Approach and Ho-Lee Model

Lectures 17 and 18 focused on modelling the spot rate \( r \). The corresponding term structures can be derived based on \( r \). This method is relatively easy to implement. A drawback is that it lacks the capability of modelling yields (or zero-coupon bond prices) with different maturities. For example, if one is interested in the spread between short and long term interest rates, those spot rate models would have difficulties to model such a feature explicitly.

An alternative approach, proposed by Heath, Jarrow and Morton (HJM), is to consider an entire yield curve as a state variable that evolves over time. In principle, one can model any one of the three equivalent term structures: zero-coupon bond prices, yields and forward rates. It is often convenient to work with forward rates. In this lecture, we start with the basic notation in HJM set-up, followed by more detailed discussion on a special case of HJM — the Ho-Lee model, and show how some spot rate models are derived from Ho-Lee model.

19.1 HJM setting

Following the basic framework in Lecture 15, we extend the idea of binomial trees to the state space consisting of yield curves. A given value of the time \( t \) term structure \( \{f(t, \tau), t \leq \tau \leq T-1\} \) may move “up” with a (risk neutral) probability \( q \) or “down” with probability \( 1-q \) to one of the two possible values of the time \( t+1 \) term structure \( \{f(t+1, \tau), t+1 \leq \tau \leq T-1\} \). In general, \( q \) may depend on \( t \) and the value of time \( t \) state. Such a binomial tree need not be recombining. To fix the notation for a binomial tree in this context, a typical node is denoted by \((t, k)\), where \( t = 0, 1, \ldots, T-1; k = 0, 1, \ldots, 2^t - 1 \). Let \( f_{tk} \) be a possible value of the time \( t \) term structure, which is a vector with components \( f_{tk}(\tau), t \leq \tau \leq T-1 \). In particular, \( f_{t+1,2k} \) and \( f_{t+1,2k+1} \) are the two immediate “descendants” of \( f_{tk} \) in the
The risk neutral (conditional) probabilities moving from the \((t, k)\) to the next generation \((t+1, 2k)\) or \((t+1, 2k+1)\) are \(q_{tk}\) and \(1-q_{tk}\) respectively.

**Example 19.1** We use the data in Example 16.1 to illustrate the above notation. First, Table 19.1 contains forward rates calculated from Table 16.1 by using the formula (15.4).

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(f(0,0))</th>
<th>(f(0,1))</th>
<th>(f(0,2))</th>
<th>(f(1,1))</th>
<th>(f(1,2))</th>
<th>(f(2,2))</th>
<th>(Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_1)</td>
<td>0.04</td>
<td>0.045</td>
<td>0.044</td>
<td>0.06</td>
<td>0.06</td>
<td>0.07</td>
<td>0.25</td>
</tr>
<tr>
<td>(\omega_2)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
</tr>
<tr>
<td>(\omega_3)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
</tr>
<tr>
<td>(\omega_4)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
<td>(~)</td>
</tr>
</tbody>
</table>

Table 19.1: Forward rates in Example 16.1.

Next, the values \(\{f_{tk}\}\) of the term structures are given as follows:

\[
\begin{align*}
    f_{00} &= (f(0,0), f(0,1), f(0,2)) \text{ on } \{\omega_1, \omega_2, \omega_3, \omega_4\} = (0.04, 0.045, 0.044) \\
    f_{10} &= (f(1,1), f(1,2)) \text{ on } \{\omega_1, \omega_2\} = (0.06, 0.06) \\
    f_{11} &= (f(1,1), f(1,2)) \text{ on } \{\omega_3, \omega_4\} = (0.03, 0.03) \\
    f_{20} &= f(2,2) \text{ on } \{\omega_1\} = 0.07 \\
    f_{21} &= f(2,2) \text{ on } \{\omega_2\} = 0.05 \\
    f_{22} &= f(2,2) \text{ on } \{\omega_3\} = 0.04 \\
    f_{23} &= f(2,2) \text{ on } \{\omega_4\} = 0.02
\end{align*}
\]

There are two restrictions that a model builder must keep in mind in the specification of term structures of forward rates: positivity of \(f(\cdot, \cdot)\) and no arbitrage. It follows from (15.5), (15.8) and (15.11) that for \(\tau = t+2, \ldots, T\),

\[
\prod_{s=t+2}^{\tau} [1 + f(t, s-1)]^{-1} = E_Q \left[ \prod_{s=t+2}^{\tau} [1 + f(t+1, s-1)]^{-1} \right] F_t. \tag{19.1}
\]

If the left-hand side is the value at a node \((t, k)\), say, then we have

\[
\prod_{s=t+2}^{\tau} \frac{1}{1 + f_{tk}(s-1)} \tag{19.2}
= q_{tk} \prod_{s=t+2}^{\tau} \frac{1}{1 + f_{t+1,2k}(s-1)} + (1-q_{tk}) \prod_{s=t+2}^{\tau} \frac{1}{1 + f_{t+1,2k+1}(s-1)}.
\]
Example 19.2 For given \( f(t, t), \ldots, f(t, T - 1) \), consider in each of the following three cases whether it is possible to specify positive values of \( f_{t+1,2k+1}(t+1), \ldots, f_{t+1,2k+1}(T-1) \) such that (19.2) holds.

Case 1. Positive values \( f_{t+1,2k}(t+1), \ldots, f_{t+1,2k}(T-1) \) and \( q_{tk} \in (0, 1) \) are fixed;

Case 2. \( q_{tk} \in (0, 1) \) is fixed, but there is flexibility in choosing positive values \( f_{t+1,2k}(t+1), \ldots, f_{t+1,2k}(T-1) \).

Case 3. Positive values \( f_{t+1,2k}(t+1), \ldots, f_{t+1,2k}(T-1) \) are fixed, but there is flexibility in choosing \( q_{tk} \in (0, 1) \).

19.2 Ho-Lee model

The paper of Ho-Lee (Journal of Finance, 1986) proposed the first yield curve model, based on which the general HJM was developed later on. In what follows, we first present Ho-Lee model for the term structures of zero-coupon bond prices then turn it into the equivalent version for forward rates, which enables us to see Ho-Lee model explicitly as a special case of HJM setting in Section 19.1 and to connect Ho-Lee model to some short rate models.

A simple feature of Ho-Lee model is that it can be represented by a recombining binomial tree (a general HJM tree need not be recombining). Therefore, the notation in Section 19.1 is modified to be adapted to the triangular lattice in Figure 17.1.

- On the triangle lattice \( \{(t, k) : k = 0, 1, \ldots, t; t = 0, 1, \ldots, T\} \), the time \( t \) term structure \( \{B(t, \tau), t < \tau \leq T\} \) assumes values \( \{B_{tk}(\tau), t < \tau \leq T\} \).

- For \( t < \tau \leq T \), let \( s = \tau - t \) be the time to maturity, \( 1 \leq s \leq T - t \). A perturbation factor \( \eta(s) \) takes either the value \( u(s) > 1 \) (an “inflation” factor or a “up” move) with probability \( q \), or the value \( 0 < d(s) < 1 \) (a “deflation” factor or a “down” move) with probability \( 1 - q \), where the (risk neutral) probability \( q \) is assumed to be a constant. The conditions

\[
\eta(0) = 1 \tag{19.3}
\]

and

\[
q \ u(s) + (1 - q) \ d(s) = 1 \quad \forall \ 1 \leq s \leq T \tag{19.4}
\]

are satisfied.

- The evolution of time \( t \) term structure in \((t, t+1]\) follows

\[
B(t + 1, \tau) = \frac{B(t, \tau)}{B(t, t+1)} \ \eta(\tau - t - 1), \tag{19.5}
\]
which can be spelled out as
\[ B_{t+1,k+1}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+1)} \ u(s-1), \] (19.6)
or
\[ B_{t+1,k}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+1)} \ d(s-1). \] (19.7)

- For \( t + 2 \leq \tau \), a “up-down” combination gives rise to
\[ B_{t+2,k+1}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+2)} \ u(s-1) \ d(s-2); \] (19.8)
while a “down-up” combination gives rise to
\[ B_{t+2,k+1}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+2)} \ d(s-1) \ u(s-2) \ d(1). \] (19.9)

Therefore, the following “path-independent” condition that forces a recombining binomial tree is needed:
\[ u(s-1) \ d(s-2) \ d(1) = d(s-1) \ u(s-2) \ u(1). \] (19.10)

Using (19.4), we can simplify (19.10) as
\[ \frac{1}{u(s-1)} = \frac{\theta}{u(s-2)} + q(1-\theta) \quad \forall \ 2 \leq s \leq T, \] (19.11)
where \( \theta \) is a constant, called the interest rate spread and given by
\[ q + (1-q)\theta = \frac{1}{u(1)}; \] (19.12)
thus \( 0 < \theta < 1 \). It follows from the induction and the condition \( u(0) = 1 \) that for \( 0 \leq s \leq T \),
\[ u(s) = \frac{1}{q + (1-q)\theta^s}, \] (19.13)
and
\[ d(s) = \frac{\theta^s}{q + (1-q)\theta^s}. \] (19.14)

- To sum up, the dynamics of term structures in Ho-Lee model is determined by two constant parameters \( q \) and \( \theta \), which makes Ho-Lee model computationally efficient but also too restrictive (underparametrized). For instance, the zero-coupon bond price may not always fall in the range \([0, 1]\). Still, Ho-Lee model plays a significant role in modelling the entire yield curve. The state at node \((t, k)\) (the time \( t \) yield curve) \((B_{tk}(t+1), B_{tk}(t+2), \ldots, B_{tk}(T))\) will change to \((B_{t+1,k+1}(t+1), \ldots, B_{t+1,k+1}(T))\) at node \((t+1, k+1)\) with probability \( q \), or \((B_{t+1,k}(t+1), \ldots, B_{t+1,k}(T))\) at node \((t+1, k)\) with probability \( 1 - q \).
Note the relation
\[
\frac{1}{1 + f(t, \tau)} = \frac{B(t, \tau + 1)}{B(t, \tau)}. \tag{19.15}
\]

Following (19.6), (19.7), (19.13) and (19.14), the dynamics of forward rates is given by
\[
\frac{1}{1 + f_{t+1,k+1}(\tau)} = \frac{1}{1 + f_{tk}(\tau)} \frac{q + (1 - q)\theta^{\tau-t-1}}{q + (1 - q)\theta^{r-t}}, \tag{19.16}
\]
or
\[
\frac{1}{1 + f_{t+1,k}(\tau)} = \frac{1}{1 + f_{tk}(\tau)} \frac{\theta [q + (1 - q)\theta^{r-t-1}]}{q + (1 - q)\theta^{r-t}}. \tag{19.17}
\]

The state \((f_{tk}(t), f_{tk}(t+1), \ldots, f_{tk}(T-1))\) will turn to \((f_{t+1,k+1}(t+1), \ldots, f_{t+1,k+1}(T-1))\) with probability \(q\), or \((f_{t+1,k+1}(t+1), \ldots, f_{t+1,k+1}(T-1))\) with probability \(1 - q\). Compare this to Section 19.1, it can be seen clearly that Ho-Lee is a special case of HJM. In particular, Ho-Lee corresponds to a recombining binomial tree, and assumes a constant probability \(q\) at every node.

**19.3 Spot rate models from Ho-Lee**

In a binomial tree associated with Ho-Lee model, the recombining feature implies that the total number of “up’s” (or “down’s”) determines the dynamics, regardless of their orders. Iterating (19.16) and (19.17) in \(t\) leads to
\[
\frac{1}{1 + f_{t,n}(\tau)} = \frac{\theta^{t-n_t}}{1 + f_{0,\tau}} \frac{q + (1 - q)\theta^{r-t}}{q + (1 - q)\theta^{r-t}}, \tag{19.18}
\]
where \(n_t\) represents the total number of “up’s” up to \(t\). Set \(\tau = t\) and recall (15.8), we have
\[
\tau(t + 1) = [1 + f_{0,t}] (q\theta^{-t} + 1 - q) \theta^{n_t} - 1. \tag{19.19}
\]

The only random variable in (19.19) is \(n_t\) which follows a binomial distribution.
Chapter 20

Forward Risk Adjusted Probability Measures

Starting from this lecture, we will discuss valuation of interest rate derivatives. For a $\mathcal{F}_\tau$-measurable contingent claim $Y(\tau)$, its time $t$ value is given by

$$V(t) = E_Q[Y(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] \quad (20.1)$$

with a risk neutral probability measure $Q$. (20.1) is particularly convenient when the bank account process $B$ is deterministic, because it simply becomes

$$V(t) = B(t)/B(\tau) \ E_Q[Y(\tau) \mid \mathcal{F}_t]. \quad (20.2)$$

In the case of stochastic interest rate $r$, an alternative method is developed based on the following change of measures.

For a fixed $\tau \leq T$, let $M(\tau) > 0$ be a $\mathcal{F}_\tau$-measurable random variable satisfying $E_Q M(\tau) = 1$. Define a new probability measure $P^\tau$ by

$$P^\tau(\omega) = M(\tau; \omega) \ Q(\omega) \ \forall \omega \in \Omega. \quad (20.3)$$

Obviously $P^\tau$ is a probability measure and $P^\tau(\omega) > 0$ for all $\omega$. Let $E^\tau(\cdot)$ denote the expectation corresponding to $P^\tau$. Define a $Q$-martingale $M = \{M(t) : t = 0, 1, \ldots, \tau\}$ by

$$M(t) = E_Q[M(\tau) \mid \mathcal{F}_t].$$

Now we state a basic result of changing martingale measures.

**Proposition 20.1** $MY = \{M(t)Y(t) : t = 0, 1, \ldots, \tau\}$ is a $Q$-martingale if and only if $Y = \{Y(t) : t = 0, 1, \ldots, \tau\}$ is a $P^\tau$-martingale.
Proof  A major step is to show that

\[ E^\tau[M(t)Y(\tau) \mid \mathcal{F}_t] = E_Q[M(\tau)Y(\tau) \mid \mathcal{F}_t] \quad \forall \ t \leq \tau. \]  

(20.4)

To verify (20.4), take an arbitrary event \( A \in \mathcal{F}_t \). Then \( M(t) \) is constant on \( A \) and

\[ M(t; \omega) = E_Q[M(\tau) \mid A] = \sum_{\omega' \in A} M(\tau; \omega') Q(\omega') / Q(A) = P^\tau(A) / Q(A) \quad \forall \ \omega \in A. \]

Therefore,

\[
E^\tau[M(t)Y(\tau) \mid A] = M(t) E^\tau[Y(\tau) \mid A]
= P^\tau(A) / Q(A) \sum_{\omega \in A} Y(\tau; \omega) M(\tau, \omega) Q(\omega) / P^\tau(A)
= \sum_{\omega \in A} Y(\tau; \omega) M(\tau, \omega) Q(\omega) / Q(A)
= E_Q[M(\tau)Y(\tau) \mid A].
\]

Note that \( MY \) is a \( Q \)-martingale if and only if \( M(t)Y(t) = E_Q[M(\tau)Y(\tau) \mid \mathcal{F}_t] \) for all \( t \leq \tau \), which is equivalent to \( Y(t) = E^\tau[Y(\tau) \mid \mathcal{F}_t] \) for all \( t \leq \tau \) by (20.4), i.e. \( Y \) is a \( P^\tau \)-martingale. This completes the proof of Proposition 20.1.

To apply Proposition 20.1 to term structure models, we first let

\[ M(\tau) = \frac{1}{B(0, \tau) B(\tau)}. \]  

(20.5)

Note that \( E_Q[M(\tau)] = E_Q[1 / B(0, \tau)] = 1. \) Hence

\[ M(t) = \frac{E_Q[1 / B(\tau) \mid \mathcal{F}_t]}{B(0, \tau)} = \frac{B(t, \tau)}{B(0, \tau) B(t)}. \]  

(20.6)

Next, define the forward risk adjusted probability measure (or called the \( \tau \)-forward measure)

\[ P^\tau(\omega) = M(\tau; \omega) Q(\omega) = \frac{Q(\omega)}{B(0, \tau) B(\tau; \omega)} \quad \forall \ \omega \in \Omega. \]  

(20.7)

Let \( S(t) \) denote the time \( t \) price of a security (e.g. stock, bond, or contingent claim). Based on (13.3), define

\[ Y(t) = FO(t) = \frac{S(t)}{E_Q[B(t)/B(\tau) \mid \mathcal{F}_t]} = \frac{S(t)}{B(t, \tau)}, \]  

(20.8)
which is the time $t$ forward price of the security to be delivered at maturity $\tau$. Hence

$$M(t)Y(t) = \frac{B(t, \tau)}{B(0, \tau)} \frac{S(t)}{B(t, \tau)} = \frac{S^\tau(t)}{B(0, \tau)},$$

which is the time $t$ discounted price of the security divided by a constant. Therefore, the process $MY$ is a $Q$-martingale. By Proposition 20.1, we obtain

**Proposition 20.2** Under the $\tau$-forward measure $P^\tau$ defined by (20.7), the forward price process $Y(\cdot) = FO(\cdot)$ with delivery time $\tau$ is a martingale. Moreover,

$$S(t) = B(t, \tau) E^\tau[S(\tau) \mid F_t]. \quad (20.9)$$

The formulas (20.9) and (20.1) will be used for pricing various interest rate derivatives in the next couple of lectures.

**Exercises:**

20.1 In Example 16.1, calculate the forward risk adjusted measures $P^\tau$ for $\tau = 1, 2, 3$.

20.2 In the binomial tree model (Lecture 5), find the expression for $P^\tau$ with $\tau = T$, and check if (20.9) agrees with (5.2).
Chapter 21

Bond Options and Coupon Bonds

Formula (20.1) or (20.9) can be used for valuation of various interest rate derivatives.

To illustrate the computation, consider Example 16.1 again, in which \( Q(\omega_i) = \frac{1}{4} \) for \( i = 1, \ldots, 4 \), and the forward risk adjusted probability measures \( P^\tau, \tau = 1, 2, 3 \) are calculated.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( B(0, 1) )</th>
<th>( B(0, 2) )</th>
<th>( B(1, 2) )</th>
<th>( B(0, 3) )</th>
<th>( B(1, 3) )</th>
<th>( B(2, 3) )</th>
<th>( P^1 )</th>
<th>( P^2 )</th>
<th>( P^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 )</td>
<td>0.9615</td>
<td>0.9203</td>
<td>0.9434</td>
<td>0.8811</td>
<td>0.8901</td>
<td>0.9346</td>
<td>0.25</td>
<td>0.2464</td>
<td>0.2405</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>0.9524</td>
<td>( \sim )</td>
<td>0.2464</td>
<td>0.2451</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>0.9709</td>
<td>( \sim )</td>
<td>0.9427</td>
<td>0.9615</td>
<td>( \sim )</td>
<td>0.2536</td>
<td>0.2547</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>( \sim )</td>
<td>0.9804</td>
<td>( \sim )</td>
<td>0.2536</td>
<td>0.2597</td>
</tr>
</tbody>
</table>

Table 20.1: Data for Example 16.1 with \( P^\tau \) added

Example 21.1 (Bond options.)

Consider an European call option on zero-coupon bond with the payoff \( [B(\tau, s) - c]^+ \). Note that \( \tau \) is the maturity of the option while \( s > \tau \) is the maturity of the zero-coupon bond. Set \( \tau = 2, s = 3 \) and the exercise price \( c = 0.96 \). Then the time \( t \) price \( S(t) \) with \( t = 1 \) is given by

\[
S(1) = B(1, 2) E^2 \{ [B(2, 3) - 0.96]^+ | F_1 \} \\
= B(1, 2) \cdot 0.2536 \cdot [(0.9615 - 0.96) + (0.9804 - 0.96)].
\]

Hence on \( \{ \omega_1, \omega_2 \} \),

\[
S(1) = 0.9434 \cdot 0.2536 \cdot 0.0219 = 0.0052;
\]

and on \( \{ \omega_3, \omega_4 \} \),

\[
S(1) = 0.9709 \cdot 0.2536 \cdot 0.0219 = 0.0054.
\]
If we use (20.1) instead, then

\[ S(1) = E_Q \{ (B(2, 3) - 0.96)^+ B(1)/B(2) \mid \mathcal{F}_1 \}. \]

Hence on \( \{\omega_1, \omega_2\} \),

\[ S(1) = 0.25 \frac{0.0219}{1.06} = 0.0052; \]

and on \( \{\omega_3, \omega_4\} \),

\[ S(1) = 0.25 \frac{0.0219}{1.03} = 0.0053. \]

Notice the discrepancy due to rounding errors. The two methods should yield the same result. In fact, Proposition 20.1 can be verified directly in this case by observing that \( B(1, 2) = B(1) B(0, 2) \).

Other bond options can be priced in the same way.

**Example 21.2** (*Coupon bonds.*)

A coupon bond is a contract entered at time \( t \), say, such that a sequence of coupons of \( c_1, \ldots, c_N \) dollars will be paid at times \( t < t_1 < \cdots < t_N \leq T \). By (20.1), the time \( t \) price of the coupon bond can be expressed as a linear combination of \( N \) zero-coupon bond prices:

\[ S(t) = \sum_{n=1}^{N} c_n B(t, t_n). \tag{21.1} \]

Now use the data in Example 21.1 and consider an European put option with maturity \( \tau = 2 \), exercise price \( c = 2.25 \), and based on the coupon bond with \( t_1 = 2, \ t_2 = 3, \ c_1 = 1.1 \) and \( c_2 = 1.2 \). The time \( t = 1 \) price of the option is

\[ S(1) = B(1, 2) E^2 \{ [c - c_1 B(2, 2) - c_2 B(2, 3)]^+ \mid \mathcal{F}_1 \}. \]

Hence on \( \{\omega_1, \omega_2\} \),

\[ S(1) = 0.9434 \cdot 0.2464 \cdot (0.02848 + 0.00712) = 0.0083; \]

and on \( \{\omega_3, \omega_4\} \),

\[ S(1) = 0.9709 \cdot 0.2464 \cdot (0.02848 + 0.00712) = 0.0085. \]

If we use (20.1), then

\[ S(1) = E_Q \{ [c - c_1 B(2, 2) - c_2 B(2, 3)]^+ B(1)/B(2) \mid \mathcal{F}_1 \}. \]
Hence on \( \{ \omega_1, \omega_2 \} \),

\[
S(1) = 0.25 \frac{0.02848 + 0.00712}{1.06} = 0.0084;
\]

and on \( \{ \omega_3, \omega_4 \} \),

\[
S(1) = 0.25 \frac{0.02848 + 0.00712}{1.03} = 0.0086.
\]
Chapter 22

Swaps, Caps and Floors

A few more interest rate derivatives are studied in this lecture.

22.1 Swaps and swaptions

A swap is an agreement between a payer and a receiver. The payer pays a fixed rate $\kappa$ to, and meanwhile receives a floating rate $r$ from the receiver. Their payments are based on a common principal. The payment is made each period during a time interval. A swap is said to be settled in arrears if one uses the spot rate $r$ for the period just ended, or settled in advance if one uses the spot rate $r$ for the period about to begin. With an ordinary swap the initial floating rate payment is based on the spot rate at the time when the agreement is made. With a forward start swap the initial floating rate payment is based on the spot rate subsequent to the one when the agreement is made. We will mainly focus on payer forward start swap on principal one settled each period in arrears. Other cases can be handled similarly. The value of a swap is the expected present value of the net cash flow, i.e.

$$S(t) = E_Q \left\{ \sum_{s=\tau}^{\tau'} [r(s) - \kappa] \frac{B(t)}{B(s)} \bigg| \mathcal{F}_t \right\}, \quad t < \tau' \leq T. \quad (22.1)$$

This formula can be simplified as follows:

$$S(t) = E_Q \left\{ \sum_{s=\tau}^{\tau'} \left[ \frac{1}{B(s-1, s)} - 1 - \kappa \right] \frac{B(t)}{B(s)} \bigg| \mathcal{F}_t \right\}$$

$$= E_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s)B(s-1, s)} \bigg| \mathcal{F}_t \right] - (1 + \kappa) E_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s)} \bigg| \mathcal{F}_t \right]$$

$$= E_Q \left[ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s-1)} \bigg| \mathcal{F}_t \right] - (1 + \kappa) \sum_{s=\tau}^{\tau'} B(t, s)$$
\[ \tau' \sum_{s=\tau}^\tau B(t, s - 1) - (1 + \kappa) \sum_{s=\tau}^\tau B(t, s) = B(t, \tau - 1) - \kappa \sum_{s=\tau}^{\tau-1} B(t, s) - (1 + \kappa) B(t, \tau'). \] (22.2)

The **forward swap rate** is the value of \( \kappa \) in (22.1) and (22.2) which makes the time \( t \) value of the forward swap in \([\tau, \tau']\) zero, denoted by

\[ \kappa(t, \tau, \tau') = \frac{B(t, \tau - 1) - B(t, \tau')}{B(t, \tau) + \cdots + B(t, \tau')} . \] (22.3)

A **payer swaption** is a European call option on the time \( \tau - 1 \) value of the corresponding payer forward start swap, with exercise time \( \tau - 1 \) and exercise price zero. For \( t < \tau \), the time \( t \) value of the payer swaption is

\[
V_p(t) = EQ \left\{ \frac{B(t)}{B(\tau - 1)} \left( EQ \left[ \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} (r(s) - \kappa) \right] \mid F_{\tau-1} \right) \mid F_t \right\} .
\]

A **receiver swaption** is defined similarly with its time \( t \) value

\[
V_r(t) = EQ \left\{ \frac{B(t)}{B(\tau - 1)} \left( EQ \left[ \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} (\kappa - r(s)) \right] \mid F_{\tau-1} \right) \mid F_t \right\} .
\]

Note that

\[
V_p(t) - V_r(t) = EQ \left\{ \frac{B(t)}{B(\tau - 1)} \left( EQ \left[ \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} (r(s) - \kappa) \right] \mid F_{\tau-1} \right) \mid F_t \right\} = EQ \left\{ \frac{B(t)}{B(\tau - 1)} \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} (r(s) - \kappa) \mid F_t \right\} = EQ \left\{ \sum_{s=\tau}^{\tau'} \frac{B(t)}{B(s)} (r(s) - \kappa) \mid F_t \right\} ,
\] (22.4)

which is the time \( t \) price of the forward start swap. Thus we have the following parity:

“payer swaption - receiver swaption = forward swap.”

By (22.2), another expression for the time \( t \) price of the payer swaption is

\[
V_p(t) = EQ \left\{ \frac{B(t)}{B(\tau - 1)} \left[ 1 - \kappa \sum_{s=\tau}^{\tau'-1} B(\tau - 1, s) - (1 + \kappa) B(\tau - 1, \tau') \right] ^+ \mid F_t \right\} ,
\]

which can be interpreted as this:
A payer swaption is same as a put option on a coupon bond, with exercise date \( \tau - 1 \) and exercise price one. The coupon has face value one and coupon rate \( \kappa \). Similarly, a receiver swaption can be interpreted as a call option on a coupon bond.

To have another interpretation, suppose for each \( s = \tau, \tau + 1, \ldots, \tau' \), there is a call option with exercise time \( s \) and payoff \( [\kappa(\tau - 1, \tau, \tau') - \kappa]^+ \). The time \( \tau - 1 \) value of this portfolio consisting of a sequence of calls is given by

\[
S(t) = E_Q \left\{ \frac{B(t)}{B(\tau - 1)} \left[ \sum_{s=\tau}^{\tau'} \frac{B(\tau - 1)}{B(s)} \left( \kappa(\tau - 1, \tau, \tau') - \kappa \right)^+ \right] \bigg| \mathcal{F}_{\tau - 1} \right\} \bigg| \mathcal{F}_t \right\}
\]

which is just the time \( t \) price of the payer swaption.

### 22.2 Caps and floors

A *caplet* is a European call option on the spot rate \( r(\tau) \) at a specific time \( \tau \) and with a specified exercise price \( \kappa \). Thus its time \( t \) price is

\[
S(t) = E_Q \{ [r(\tau) - \kappa]^+ B(t)/B(\tau) \mid \mathcal{F}_t \} = B(t, \tau) \ E^{\tau} \{ [r(\tau) - \kappa]^+ \mid \mathcal{F}_t \}.
\]

A *cap* is a sequence of caplets which have a common exercise price. Like swaps, there are ordinary and forward start caps, depending on whether the initial caplet corresponds to the current spot rate. The time \( t \) price of a forward start cap in \([\tau, \tau']\) settled in arrears is given by

\[
S(t) = \sum_{s=\tau}^{\tau'} E_Q \{ [r(s) - \kappa]^+ B(t)/B(s) \mid \mathcal{F}_t \}, \quad t < \tau \leq \tau' \leq T. \tag{22.6}
\]

A *floorlet* is defined in the same way as a caplet, but as a put option on the spot rate. A *floor* is a sequence of floorlets. The time \( t \) price of a forward start floor in \([\tau, \tau']\) settled in arrears is given by

\[
S(t) = \sum_{s=\tau}^{\tau'} E_Q \{ [\kappa - r(s)]^+ B(t)/B(s) \mid \mathcal{F}_t \}, \quad t < \tau \leq \tau' \leq T. \tag{22.7}
\]
It is obvious that the price of a cap minus the price of a floor is equal to the price of a swap.

Furthermore, a caption is a call or put option on a forward cap; while a floortion is a call or put option on a forward floor. These derivatives are examples of compound options.
Chapter 23

Implied Volatility

Financial models need to be fitted by real data in financial markets. This is called model calibration — a major task in empirical finance. An important special case is volatility estimation. We start with some motivation then introduce a couple of approaches along this line.

23.1 Inverting the Black-Scholes formula

Recall the Black-Scholes formula (6.7) for a European call option price:

\[ C = C(t, T) = S(t) \Phi(v_1) - c e^{-r(T-t)} \Phi(v_2), \]

where

\[ v_1 = \frac{\log(S(t)/c) + (r + \sigma^2/2) (T-t)}{\sigma \sqrt{T-t}} \]

and

\[ v_2 = v_1 - \sigma \sqrt{T-t} = \frac{\log(S(t)/c) + (r - \sigma^2/2) (T-t)}{\sigma \sqrt{T-t}}. \]

Notice that \( t, T, c \) and \( S(t) \) are clearly known to investors, but the parameters \( r \) and \( \sigma \) need to be specified when applying formula (6.7). A more basic question is whether it is reasonable to assume that \( r \) and \( \sigma \) are constants. In fact, a lot of efforts have been made to challenge and modified the basic assumptions for the Black-Scholes theory:

(i) constant parameters \( r \) and \( \sigma \);

(ii) geometric Brownian motions as models for the underlying risky security \( \{S(t)\} \).
We skip the issue (ii) since this course is only limited to discrete-time models. For the issue (i), the case of stochastic spot rate $r$ was studied in some previous lectures. Our focus now is on modification of constant volatility $\sigma$.

A statistical interpretation of volatility $\sigma$ dates back to the sequence of binomial trees in Lecture 6. For each positive integer $n$ and fixed $T > 0$, we let the up and down factors be

$$\begin{align*}
    u_n &= e^{a_n} (1 + r_n) \\
    d_n &= e^{-a_n} (1 + r_n)
\end{align*}$$

where $a_n = \sigma \sqrt{\delta}$ and $r_n = r\delta$ with two constant parameters $r > 0$, $\sigma > 0$, and $\delta = T/n$.

For large $n$ (or small $\delta$), we can rewrite the two factors as

$$\begin{align*}
    \log u_n &\approx r\delta + \sigma \sqrt{\delta} \\
    \log d_n &\approx r\delta - \sigma \sqrt{\delta}
\end{align*}$$

where the LHS represent a one-step return with step size $\delta$. Thus $r\delta$ is thought of as the mean return, and $\sigma \sqrt{\delta}$ the standard deviation.

This interpretation motivates a standard statistical method to estimate volatility $\sigma$, often referred to as historical volatility, in which the data consist of observations of returns, and the corresponding sample standard deviation is calculated as an estimate of volatility $\sigma$. However, a different approach is used more often in daily practice of financial markets.

The basic strategy is to derive an estimate of $\sigma$ by “inverting” a pricing formula such as (6.7). Does the inversion work? In the case of (6.7), it turns out that for fixed $t, T, c, r$ and $S(t), C$ is strictly increasing in $\sigma$. To verify this, notice that

$$\frac{dv_1}{d\sigma} - \frac{dv_2}{d\sigma} = \sqrt{T - t}; \quad (23.1)$$

and

$$S(t) \phi(v_1) = c \phi(v_2) e^{-r(T - t)}, \quad (23.2)$$

where $\phi$ is the density function of $N(0,1)$. Therefore,

$$\begin{align*}
    \frac{\partial C}{\partial \sigma} &= S(t) \phi(v_1) \frac{dv_1}{d\sigma} - c \left( e^{-r(T - t)} \phi(v_2) \frac{dv_2}{d\sigma} \
    &= S(t) \phi(v_1) \left( \frac{dv_1}{d\sigma} - \frac{dv_2}{d\sigma} \right) \quad \text{[by (23.2)]} \\
    &= S(t) \phi(v_1) \sqrt{T - t} \quad \text{[by (23.1)]} \\
    &> 0. \quad (23.3)
\end{align*}$$
As the result of inverting (6.7),

\[ \sigma = \sigma(C; t, T, c, r, S(t)) \]

is referred to as the \textit{implied volatility}.

\textbf{Note:}

In general, a pricing formula need not be invertible. If it is invertible, then a corresponding implied volatility can be derived. Hence

an implied volatility depends on a particular pricing formula, while the historical volatility aforementioned does not. In that sense, an implied volatility reflects the market variability affected by certain specific securities, while the historical volatility is a measure of overall market variability or uncertainty.

\section*{23.2 Volatility smile}

Options are referred to as \textit{in-the-money}, \textit{at-the-money}, or \textit{out-of-the-money}. An in-the-money option is one that would lead to a positive cash flow to the holder if it were exercised immediately; similarly, an at-the-money option would lead to zero cash flow while an out-of-the-money option a negative cash flow. For a call option, the three cases correspond to \( S(\tau) > c \), \( S(\tau) = c \) and \( S(\tau) < c \), respectively, at the expiry \( \tau \). Observed implied volatilities often reveal a “smile”, which caricatures a nonlinear shape of an implied volatility plot against the exercise price \( c \).

(See the attached figure.)

The \textit{volatility smile} effect is a manifestation of violation of constant volatility assumption in the Black-Scholes theory. Various factors in a financial market, such as exercise price \( c \), affect the volatility. Even if the Black-Scholes model is slightly modified by allowing the volatility to depend on the exercise price, i.e. \( \sigma = \sigma(c) \), it still cannot explain the volatility smile. To see this, apply the Implicit Function Theorem to the formula (6.7), we have

\[
\frac{d\sigma}{dc} = -\frac{\frac{\partial C}{\partial c}}{\frac{\partial C}{\partial \sigma}}.
\]

By (23.3), \( \frac{d\sigma}{dc} \) and \( \frac{\partial C}{\partial c} \) have opposite signs. By (23.2),

\[
\frac{\partial C}{\partial c} = S(t) \phi(v_1) \frac{dv_1}{dc} - c e^{-r(T-t)} \phi(v_2) \frac{dv_2}{dc} - e^{-r(T-t)} \Phi(v_2)
\]

\[
= - e^{-r(T-t)} \Phi(v_2)
\]

\[
< 0.
\]
Hence $\frac{d\sigma}{dc} > 0$, i.e. $\sigma$ is strictly increasing in $c$, which does not agree with the volatility smile phenomenon.

A nonconstant volatility can be treated as a random variable, or a deterministic function in time, or a stochastic process. The study of various stochastic volatility models has become an active research area in which volatility itself is defined as a stochastic process.

Notes:

- Apart from volatility smile, another component of implied volatility is the term structure of volatility which describes the way at-the-money implied volatility varies with time to expiration.

- Although in a stochastic volatility setting, volatility can be modelled in the same manner as stocks and bonds, volatility is neither an observable variable nor a tradable security. This brings up an important issue of hedging volatility risk — the risk associated with unanticipated changes in volatility. If there is a security, called volatility instrument, such that its price is perfectly correlated with volatility changes, then volatility risk can be hedged out through this financial instrument. However, such a situation is rare. In other words, a stochastic volatility model is an incomplete market in general.
Chapter 24

Implied Volatility Trees

The basic idea of implied volatility trees, introduced by Derman and Kani, is to build a binomial tree model for underlying securities with a given set of option prices as its input. It is designed so that (i) the option prices derived from the implied tree match the input consistently; (ii) certain parameters, such as volatility, can be inferred from the implied tree; and (iii) more exotic options and other derivatives can be priced via the implied tree.

24.1 Construction of implied volatility trees via forward induction

An implied volatility tree is a recombining binomial tree of security prices on the triangle lattice \{(t, k) : k = 0, 1, \ldots, t; \ t = 0, 1, \ldots, T\} (same as in Lecture 17 or Section 19.2). The lattice could be renormalized by introducing a temporal increment \(\delta = T/n\) and a spatial increment \(\sqrt{\delta}\) as we did before in order to make a connection to some continuous-time models, but we choose not doing it to keep the notation simple.

Using the forward induction, suppose the tree has been built up to time \(t\). For a fixed node \((t, k)\), we will show how to derive the security prices at \((t + 1, k)\) and \((t + 1, k + 1)\) based on the security price at \((t, k)\) and some option prices at time \(t\). In particular, assume the knowledge of time \(t\) values of some European put options.

In general, each variable and parameter of interest can depend on time and state. We adopt the following simplified notation:

- Up and down factors: \(u = u(t, k)\) from \((t, k)\) to \((t + 1, k + 1)\) and \(d = d(t, k)\) from \((t, k)\) to \((t + 1, k)\) respectively.

- Security prices: \(S = S(t, k)\), \(S_u = S(t+1, k+1) = S\ u\) and \(S_d = S(t+1, k) = S\ d\). Since the tree is recombining, \(S_d\) for \((t, k)\) is also \(S_u\) for \((t, k - 1)\), and vice versa.
• Short rate in \((t, t + 1)\): \(r = r(t + 1)\) (recall that the short rate process is assumed to be predictable).

• Risk neutral probability: \(q = q(t, k) = \frac{1 + r - d}{u - d} = \frac{(1 + r)S - S_d}{S_u - S_d}\).

• The value of a European put at \((t, k)\): \(V = V_{\text{put}}(t, k)\) with expiry \(t + 1\) and strike price \(c = S = S(t, k)\) (at-the-money put). This is equivalent to the knowledge of the European call \(V_{\text{call}}(t, k)\) with the same expiry \(t + 1\) and same strike price \(c = S(t, k)\) because of the put-call parity.

Having built the tree up to time \(t\), there are \(t + 2\) variables \(S(t + 1, k)\), \(k = 0, 1, \ldots, t + 1\) to be specified. We establish \(t + 2\) constraints with the following steps.

(i) Each node at time \(t\) has an at-the-money option pricing constraint

\[
V = (1 + r)^{-1}(1 - q) (S - S_d),
\]

with the strike price \(c = S\), where

\[
1 - q = \frac{S_u - (1 + r)S}{S_u - S_d}.
\]

This amounts to \(t + 1\) constraints.

(ii) An additional constraint can be obtained by imposing the symmetry condition \(d = 1/u\) at a particular node near the “center”, which is identified as \((t, t/2)\) if \(t\) is even or \((t, (t + 1)/2)\) if \(t\) is odd. From this central node \((t, k)\), we compute the up factor as

\[
u = \frac{V + S}{(1 + r)^{-1}S - V},
\]

and the down factor \(d = 1/u\), then obtain \(S_u\) and \(S_d\). To verify (24.3), plugging (24.2) in (24.1) and reorganizing terms yields the equation

\[
[(1 + r)^{-1}S - V] u - S [1 + (1 + r)^{-1}] + \frac{S + V}{u} = 0,
\]

which is equivalent to the quadratic equation

\[
[(1 + r)^{-1}S - V] u^2 - S [1 + (1 + r)^{-1}] u + (S + V) = 0,
\]

thus

\[
u = \frac{S [1 + (1 + r)^{-1}] \pm \{S[1 - (1 + r)^{-1}] + 2V\}}{2 [(1 + r)^{-1}S - V]}.
\]

We choose “+” since there is no guarantee that “−” will lead to a positive value of \(u\). This will result in (24.3).
Having calculated $S_u = S(t+1, k+1)$ and $S_d = S(t+1, k)$ from the central node $(t, k)$, we compute security prices at nodes $(t+1, j)$ with $j > k+1$ and $j < k$. Following the recombining property, $S_u$ for $(t, k)$ is also $S_d$ for $(t, k+1)$, and $S_d$ for $(t, k)$ is also $S_u$ for $(t, k-1)$. Therefore, using

$$S_u = \frac{V S_d + S[(1+r)^{-1}S_d - S]}{V + (1+r)^{-1}S_d - S}, \quad (24.6)$$

and

$$S_d = \frac{V S_u + S[S - (1+r)^{-1}S_u]}{V + S - (1+r)^{-1}S_u}, \quad (24.7)$$

security prices at all nodes at time $t+1$ can be computed.

### 24.2 Specification of $V_{\text{put}}$ via Arrow-Debreu securities

In the construction of an implied volatility tree, the option price $V = V_{\text{put}}(t, k)$ corresponding to the strike price $c = S(t, k)$ at each node $(t, k)$ is needed as an input. First, note that $V_{\text{put}}(t, j) = 0$ for all $j > k$. Second, for $j < k$, we have

$$V_{\text{put}}(t, j) = \frac{S(t, k)}{1+r(t+1)} - S(t, j). \quad \text{(Why?)}$$

Third, the data available in an option market is usually not $V_{\text{put}}(t, k)$ at various nodes $(t, k)$, but their averages. Let $P(t, \tau, c)$ denote the time $t$ market price of a European put option with expiry $\tau$ and strike price $c$. Then

$$P(t, t+1, S(t, k)) = \sum_{j=0}^{k} p_{t,j} V_{\text{put}}(t, j)$$

$$= \sum_{j=0}^{k-1} p_{t,j} \left[ \frac{S(t, k)}{1+r(t+1)} - S(t, j) \right] + p_{t,k} V_{\text{put}}(t, k), \quad (24.8)$$

which gives rise to

$$V_{\text{put}}(t, k) = \frac{P(t, t+1, S(t, k)) - \sum_{j=0}^{k-1} p_{t,j} \left[ \frac{S(t, k)}{1+r(t+1)} - S(t, j) \right]}{p_{t,k}}, \quad (24.9)$$

where $p_{t,j}$ is the probability (under the EMM $Q$) that the security attains the value $S(t, j)$.

In this calculation, the known market price $P(t, t+1, S(t, k))$ is represented as an average of $V_{\text{put}}(t, j)$ weighted by Arrow-Debreu probabilities $\{p_{t,j}\}$, which can be calculated.
iteratively by
\[
\begin{align*}
  p_{t+1,0} &= [1 - q(t, 0)] p_{t,0}, \\
p_{t+1,t+1} &= q(t, t) p_{t,t}, \\
p_{t+1,k} &= q(t, k-1) p_{t,k-1} + [1 - q(t, k-1)] p_{t,k}, \quad 1 \leq k \leq t.
\end{align*}
\] (24.10)

In particular, \( p_{0,0} = 1 \). Arrow-Debreu probabilities are related to Arrow-Debreu securities: the Arrow-Debreu security associated with each node \((t, k)\) is a contingent claim at time \( t \) which pays $1 if the state \((t, k)\) is attained by the security and $0 otherwise. Although these securities are not traded in the market, they can still be priced via the risk neutral valuation principle. Let \((t', k')\) be a node with \( t' < t \), then at \((t', k')\), the price of the Arrow-Debreu security associated with node \((t, k)\) is the sum of probabilities attached to all paths that lead from \((t', k')\) to \((t, k)\), multiplied by the discount factor \( \{[1 + r(t' + 1)] \cdots [1 + r(t)]\}^{-1} \).

### 24.3 How to deal with possible bad probabilities?

A possible complication in implementing the above procedure is that at a certain node \((t, k)\), the resulting risk neutral probability \( q(t, k) \) may turn out to be a “bad probability”, i.e. \( 0 \leq q \leq 1 \) need not be satisfied, or equivalently, the inequalities \( S_d < S < S_u \) may not hold. When that happens, we override the option price \( V \) that produces the bad node, and choose the security prices that keep the local volatility
\[
\sigma(t, k) = \sqrt{q(1-q)} \log \frac{S_u}{S_d}
\]
the same as that calculated at two preceding nodes. See [1] and [2] for more detailed discussions.


**References on “implied trees”:**


Problem Set 1

(1) In Example 4.1 (same as the example in Lecture 2 except for the interest rate \( r = 7\% \)),
(1a) find an arbitrage opportunity;
(1b) show that the law of one price remains true.

(2) Construct a single period model \((T = 1)\) for which the law of one price holds but there exists an arbitrage opportunity.

(3) Do Exercise 4.3.

(4) Do Exercise 6.1.

(5) Recall the example in Lecture 2: \( u = 1.07, d = 0.92, r = 0.06, \) and the stock price follows the binomial tree in Figure 2.1.
(5a) Construct a put option tree (similar to Figure 2.2) with the exercise price \( c = \$2.05 \).
(5b) For the chooser option (see Example 5.3) with the decision time \( T_0 = 2 \), the expiry \( T = 3 \), and the exercise price \( c = \$2.05 \), construct the replicating portfolio and represent it in a binomial tree similar to Figure 2.3.
(5c) Construct a binomial tree for the chooser option values under the conditions in (5b).
Problem Set 2

(1) Consider the binomial tree model: \( T = 3, \ u = 1.07, \ d = 0.92, \ r = 6\% \) and \( S(0) = $2 \) (see the example in Lecture 2). Suppose a constant dividend yield \( \lambda = 5\% \) of the stock price is issued at the ex-dividend date \( t = 2 \).

(1a) Calculate the value process of the American chooser option with the decision time \( T_0 = 1 \) and the exercise price \( c = $2.05 \).

(1b) Determine the optimal exercise strategy.

(2) Consider the binomial tree model with parameters \( T, \ u, \ d, \ r \) and \( p \), where \( d < 1 + r < u \). Assume the initial price \( S(0) \), the initial wealth \( w_0 \) and the utility function \( U(w) = \log w, w > 0 \). Use dynamic programming to determine the optimal trading strategy and verify if your result agrees with the conclusion on Example 11.2 in Lecture 11.

(3) Consider the binomial tree model: \( T = 2, \ u = 1.07, \ d = 0.92, \ r = 4\% \) and \( S(0) = $1 \); and assume the probability \( p = 0.6 \).

(3a) Solve the problem (P4) in Lecture 12 at \( t = 1 \) and with \( \mu_1 = 0.05 \).

(3b) At \( t = 1 \), calculate the risk premium \( E_t R(t + 1) - r \) associated with a European call option with the exercise price \( c = $1 \).

(3c) Do (3b) again using the CAPM (12.1), and compare your result here with what you obtained in (3b).
Problem Set 3

(1) Assume the stock price process starts with $S(0) = 1$ and follows a binomial tree with $u = 1.08$, $d = 0.91$. Moreover, the short rate process is the same as in Example 16.1.

(1a) Consider a forward contract and a futures contract on the stock, both start at $t = 0$ and end at $T = 3$. Do they have the same value at $t = 0$? Why?

(1b) For a European call option and a European put option on the futures $FU(1, 3)$, specify the exercise price $c$ such that these two European options have the same value at $t = 0$.

(2) Consider the following numerical problem related to implied volatility. Use an iterative method, such as the Newton-Raphson method (or others if you prefer), to compute the implied volatility via the Black-Scholes formula for European call options, where we assume that the current stock price $S(t) = 100$, the strike price $c = 100$, the time to expiration $T - t = 2$ (years), the annual interest rate $r = 3\%$ and the market option price $C = 8.61$. Briefly describe the algorithm(s) you use, with the parameters in your algorithm: initial values (e.g. you may try to start with an initial volatility $\sigma = 20\%$), number of steps, etc., report your results via tables or/and plots, and make comments if necessary.

(3) Instead of an at-the-money option considered in (2), this problem involves a number of different strike prices given in the following table, in order to see whether the resulting implied volatilities demonstrate a volatility smile. Calculate the implied volatilities and plot them against the strike prices.

<table>
<thead>
<tr>
<th>Type of option</th>
<th>Strike price</th>
<th>Mean option price</th>
<th>Implied volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>90</td>
<td>4.75</td>
<td></td>
</tr>
<tr>
<td>call</td>
<td>95</td>
<td>1.375</td>
<td></td>
</tr>
<tr>
<td>call</td>
<td>100</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>put</td>
<td>90</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>put</td>
<td>95</td>
<td>2.875</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: One day option prices. The current stock price $S(t) = 93.625$; the time to expiration $T - t = 22$ (days); the annual interest rate $r = 5.12\%$; a 360-day year assumed; the mean of the bid-ask prices for each option is used as the observed market option price. See Hull’s book Chapter 6 for more detailed discussion on options markets.

(4) Do Exercise 20.2.

(5) Consider several interest rate derivatives based on the model and data given in Example 16.1.
(5a) For \( t = 0, \tau = 1 \) and \( \tau' = 2 \), find the value of \( \kappa \) such that \( V_p(t) = V_r(t) \), i.e. the payer and receiver swaptions have the same value at \( t = 0 \) (assuming the underlying swap is ordinary and settled in advance).

(5b) For \( \tau = 2 \) and \( \tau' = 3 \), find the value of \( \kappa \) such that the cap and floor have the same value at time \( t = 0 \) (both assumed to be ordinary and settled in arrears).

(5c) Consider a call caption and a put caption with the same expiration time \( \tau = 2 \) and the same exercise price \( c \), based on the ordinary cap defined in (5b) [use your result in (5b) as the value of \( \kappa \)]. Find the value of \( c \) such that the call and put captions have the same value at time \( t = 0 \).