Convergence of the long memory Markov switching model to Brownian motion

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Abstract

This technical note shows that the partial sum process of the Markov switching model converges to Brownian motion in the sense of finite-dimensional distributions.

1 Definitions

The Markov switching (MS, in short) model is defined as follows. Let \( s_T = \{s_k^T\}_{k=1,...,T} \) be a stationary Markov chain, taking the value 0 or 1, with the probability transition matrix

\[
P = \begin{pmatrix}
p_{00} & 1 - p_{00} \\
1 - p_{11} & p_{11}
\end{pmatrix}.
\]

(1.1)

We shall take below \( p_{00} = p_{00}(T) \) and \( p_{11} = p_{11}(T) \) as functions of \( T \), which explains the use of the index \( T \) in the notation \( s^T \). Let \( \mu_0, \mu_1 \in \mathbb{R} \). Consider the series

\[
X_k^T = \mu_{s_k^T} + \epsilon_k, \quad k = 1, \ldots, T,
\]

(1.2)

where \( \{\epsilon_k\} \) is a sequence of i.i.d. random variables with mean 0 and variance \( \sigma^2_{\epsilon} \), and its partial sum process

\[
S_T(t) = \sum_{k=1}^{\lfloor Tt \rfloor} (X_k^T - \mathbb{E}X_k^T), \quad t \in [0, 1].
\]

The term \( \mu_{s_k^T} \) in \( X_k^T \) is not constant. It varies between two values \( \mu_0 \) and \( \mu_1 \). The switching mechanism is dictated by the Markov chain \( s_k^T, \ k = 1, \ldots, T \). If one is in regime \( j, \ j = 0, 1 \), one stays there with probability \( p_{jj} \) and switches to the other regime with probability \( 1 - p_{jj} \).

Suppose now that

\[
\mu_0 \neq \mu_1
\]

(1.3)

and, for \( j = 0, 1 \),

\[
p_{jj} = p_{jj}(T) = 1 - \frac{c_j}{T^{\delta_j}},
\]

(1.4)

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where $0 < c_j < 1$ and

$$0 < \delta_0, \delta_1 < 1.$$  \hfill (1.5)

Since $p_{jj}(T) \to 1$, the Markov chain $s^T$ tends to stay in the same state as $T \to \infty$. Under the assumptions (1.3)–(1.5), Diebold and Inoue (2001), Proposition 3, showed that

$$\text{Var}(S_T(1)) = \text{Var}(X_1^T + \ldots + X_T^T) \sim CT^{2d+1},$$  \hfill (1.6)

where

$$d = \frac{1}{2} \left( \min\{\delta_0, \delta_1\} - |\delta_0 - \delta_1| \right) \in \left( 0, \frac{1}{2} \right).$$  \hfill (1.7)

The behavior (1.6) of the variance of the partial sum is consistent with that for long memory time series. Recall that a long memory time series $X = \{X_k\}_{k \in \mathbb{Z}}$ is commonly defined as a second-order stationary time series with autocovariance function

$$\text{Cov}(X_0, X_h) \sim ch^{2d-1}, \quad \text{as} \quad h \to \infty,$$  \hfill (1.8)

where $c > 0$ and $d \in (0, 1/2)$. Under (1.8), one can easily show that $\text{Var}(X_1 + \ldots + X_T) \sim CT^{2d+1}$, as $T \to \infty$. The parameter $d$ is known as the long memory parameter.

## 2 Convergence result

We are interested here not only in the second-order properties of the partial sum process $S_T(t)$ but also in its convergence in distribution. For a long memory time series, a common limit of its partial sum process is fractional Brownian motion with parameter $H = d + 1/2$. Recall that $B_H = \{B_H(t)\}_{t \in \mathbb{R}}$ is fractional Brownian motion (FBM) if it is Gaussian, has stationary increments and is $H$-self-similar (i.e. for any $c > 0$, the process $B_H(ct)$ and $c^HB_H(t)$ have the same finite-dimensional distributions). It is called standard if $\mathbb{E}B_H(1)^2 = 1$, and if $H = 1/2$, then it is the usual Brownian motion. Does the partial sum process $S_T(t)$ of the Markov switching model converge to FBM with parameter $H = d + 1/2$? The answer to this question is negative, as will be shown below (under additional assumptions). In fact, the limit turns out to be the usual Brownian motion. Consequences of this fact for estimation of the long memory parameter in the MS model are discussed in Baek, Fortuna and Pipiras (2014).

Before providing the convergence result, we state an auxiliary lemma which will be used in the proof of the main result. The result of the lemma is implicit in the proof of Proposition 3 in Diebold and Inoue (2001). See Appendix for a proof.

**Lemma 2.1** The term $\mu_{s_T}$ in the Markov switching model (1.2) satisfies: for $1 \leq j, k \leq T$,

$$\text{Cov} \left( \mu_{s_T}^j, \mu_{s_T}^m \right) = \mu^T \Gamma_{|k-m|} \mu = \frac{(1-p_{00})(1-p_{11})}{(2-p_{00}-p_{11})^2} (\mu_0 - \mu_1)^2 \lambda_{|k-m|},$$  \hfill (2.1)

where

$$\mu = \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix}, \quad \Gamma_j = \frac{(1-p_{00})(1-p_{11})}{(2-p_{00}-p_{11})^2} \begin{pmatrix} \lambda^j & -\lambda^j \\ -\lambda^j & \lambda^j \end{pmatrix}$$  \hfill (2.2)

and

$$\lambda = p_{00} + p_{11} - 1.$$  \hfill (2.3)

In the next result, we establish the convergence of the partial sum process to Brownian motion. See Appendix for a proof.
Theorem 2.1 Suppose that (1.3)–(1.5) hold and that
\[ 3 \min \{ \delta_0, \delta_1 \} + |\delta_0 - \delta_1| < 1. \]  \tag{2.4}

Then,
\[ \frac{S_T(t)}{T^{1/2 + d}} \xrightarrow{fdd} \sigma B(t), \]  \tag{2.5}

where \( d \) is given in (1.7), \( fdd \) stands for the convergence of finite-dimensional distributions and \( B \) is a standard Brownian motion with
\[ \sigma^2 = \frac{2c_0c_1(\mu_0 - \mu_1)^2}{(\bar{c})^3} \]  \tag{2.6}

and
\[ \bar{c} = \begin{cases} 
  c_0, & \text{if } \delta_0 < \delta_1, \\
  c_1, & \text{if } \delta_1 < \delta_0, \\
  c_0 + c_1, & \text{if } \delta_0 = \delta_1. 
\end{cases} \]  \tag{2.7}

The condition (2.4) in Theorem 2.1 was used to obtain a normal limit for the partial sum process \( S_T \). Note that, when \( \delta = \delta_0 = \delta_1 > 0 \), this condition becomes
\[ 0 < \delta < 1/3 \]

and hence (see (1.7))
\[ d = \delta_0 - \frac{\delta_1}{2} \in \left(0, \frac{1}{6}\right). \]

More generally suppose \( \delta_0 \leq \delta_1 \). Then (1.7) becomes
\[ d = \delta_0 - \frac{\delta_1}{2}. \]

The condition \( d > 0 \) translates to \( \delta_0 > \delta_1/2 \), thus \( \delta_1/2 < \delta_0 \leq \delta_1 \). Moreover, (2.4) becomes \( 2\delta_0 + \delta_1 < 1 \), that is, \( \delta_0 < (1 - \delta_1)/2 \). Two cases need to be distinguished next.

First, suppose that \( \delta_1 < (1 - \delta_1)/2 \) or \( \delta_1 < 1/3 \). Then, the two conditions \( \delta_1/2 < \delta_0 \leq \delta_1 \) and \( \delta_0 < (1 - \delta_1)/2 \) reduce to the first one, namely, \( \delta_1/2 < \delta_0 \leq \delta_1 \). In particular,
\[ d = \delta_0 - \frac{\delta_1}{2} \leq \frac{\delta_1}{2} \text{ and } d \in \left(0, \frac{1}{6}\right), \]

since \( \delta_1 < 1/3 \).

Second, suppose that \( \delta_1 \geq (1 - \delta_1)/2 \) or \( \delta_1 \geq 1/3 \). Then, the two conditions \( \delta_1/2 < \delta_0 \leq \delta_1 \) and \( \delta_0 < (1 - \delta_1)/2 \) reduce to \( \delta_1/2 < \delta_0 \leq (1 - \delta_1)/2 \) (supposing \( \delta_1 < (1 - \delta_1)/2 \) or \( \delta_1 < 1/2 \); otherwise, the intervals in the two conditions are disjoint). In particular,
\[ d = \delta_0 - \frac{\delta_1}{2} \leq \frac{1}{2} - \delta_1 \text{ and } d \in \left(0, \frac{1}{6}\right), \]

since \( \delta_1 \geq 1/3 \).

What happens when the condition (2.4) does not hold (that is, for other values of \( \delta_0, \delta_1 \))? When \( t = 1 \), the convergence to a normal limit follows immediately from the results of Dobrushin (1961). In fact, the asymptotic normality is expected for all finite-dimensional distributions. In the proof of Theorem 2.1, we used general convergence results for nonstationary Markov chains. In a "nearly stationary" case, the conditions of those results are expected to be too strong as discussed in the footnote on p. 76 of Dobrushin (1956). But to the best of our knowledge, there are currently no results addressing this issue (aside from the result corresponding to \( t = 1 \)).
A Technical proofs

Proof of Lemma 2.1: Let

$$\xi_k = \begin{pmatrix} 1_{\{s^T_k=0\}} \\ 1_{\{s^T_k=1\}} \end{pmatrix},$$

so that $$\mu_{s^T_k} = \mu' \xi_k$$, where prime denotes the transpose. Observe further that

$$\text{Cov} \left( \mu_{s^T_k}, \mu_{s^T_m} \right) = \text{Cov} \left( \mu' \xi_k, \mu' \xi_m \right) = \mu' \text{Cov}(\xi_k, \xi_m) \mu = \mu' \Gamma_{|k-m|} \mu,$$  \hspace{1cm} (A.1)

where

$$\Gamma_j = \text{Cov}(\xi_0, \xi_j) = \begin{pmatrix} \text{Cov}(1_{\{s^T_0=0\}}, 1_{\{s^T_0=0\}}) & \text{Cov}(1_{\{s^T_0=0\}}, 1_{\{s^T_1=1\}}) \\ \text{Cov}(1_{\{s^T_0=1\}}, 1_{\{s^T_0=0\}}) & \text{Cov}(1_{\{s^T_0=1\}}, 1_{\{s^T_1=1\}}) \end{pmatrix} = \begin{pmatrix} \mathbb{P}(s^T_0 = 0, s^T_j = 0) - \mathbb{P}(s^T_0 = 0) \mathbb{P}(s^T_j = 0) & \mathbb{P}(s^T_0 = 0, s^T_j = 1) - \mathbb{P}(s^T_0 = 0) \mathbb{P}(s^T_j = 1) \\ \mathbb{P}(s^T_0 = 1, s^T_j = 0) - \mathbb{P}(s^T_0 = 1) \mathbb{P}(s^T_j = 0) & \mathbb{P}(s^T_0 = 1, s^T_j = 1) - \mathbb{P}(s^T_0 = 1) \mathbb{P}(s^T_j = 1) \end{pmatrix}.$$

Denoting by $$\pi = (\pi_0, \pi_1)'$$ the invariant (stationary) probabilities of $$s^T$$, and by $$p_{nm}^j$$ the transition probability from state $$n$$ to state $$m$$ in $$j$$ steps, we have

$$\Gamma_j = \begin{pmatrix} \pi_0 (p_{00}^j - \pi_0) & \pi_0 (p_{01}^j - \pi_1) \\ \pi_1 (p_{10}^j - \pi_0) & \pi_1 (p_{11}^j - \pi_1) \end{pmatrix}. \hspace{1cm} (A.2)$$

To evaluate $$\Gamma_j$$, we need expressions for $$\pi_0$$, $$\pi_1$$ and $$p_{nm}^j$$. One can show that

$$\pi_0 = \frac{1 - p_{11}}{2 - p_{00} - p_{11}}, \quad \pi_1 = \frac{1 - p_{00}}{2 - p_{00} - p_{11}} \hspace{1cm} (A.3)$$

and that the probability transition matrix $$P$$ in (1.1) is diagonalizable as

$$P = \begin{pmatrix} 1 & \pi_1 \\ 1 & -\pi_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \pi_0 & \pi_1 \\ 1 & -1 \end{pmatrix} =: E \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} F', \hspace{1cm} (A.4)$$

where $$\lambda$$ is given by (2.3). Note that 1 and $$\lambda$$ in (A.4) are the eigenvalues of the matrix $$P$$ and the matrices $$E$$ and $$F$$ are made of the right- and left-eigenvectors of $$P$$, satisfying $$F' = E^{-1}$$. The latter fact implies that

$$P^j = \begin{pmatrix} 1 & \pi_1 \\ 1 & -\pi_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \pi_0 & \pi_1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \pi_0 + \pi_1 \lambda^j & \pi_1 (1 - \lambda^j) \\ \pi_0 (1 - \lambda^j) & \pi_1 + \pi_0 \lambda^j \end{pmatrix}$$

$$= \frac{1}{2 - p_{00} - p_{11}} \begin{pmatrix} (1 - p_{11}) + (1 - p_{00}) \lambda^j & (1 - p_{00})(1 - \lambda^j) \\ (1 - p_{11})(1 - \lambda^j) & (1 - p_{00}) + (1 - p_{11}) \lambda^j \end{pmatrix} = \begin{pmatrix} p_{00}^j & p_{01}^j \\ p_{10}^j & p_{11}^j \end{pmatrix}. \hspace{1cm} (A.5)$$

By using (A.3) and (A.5), we obtain from (A.2) that $$\Gamma_j$$ is given by (2.2) and (2.1) is finally established by using (A.1). \quad \Box

Proof of Theorem 2.1: It is enough to show that, for $$t_1, \ldots, t_n \in [0, 1]$$ and $$\theta_1, \ldots, \theta_n \in \mathbb{R},$$

$$\sum_{k=1}^{n} \theta_k \frac{S_T(t_k)}{T^{1/2+d}} \overset{d}{\rightarrow} \mathcal{N}(0, \sigma_n^2) \hspace{1cm} (A.6)$$
and that
\[ \sigma_n^2 = \lim_{T \to \infty} \mathbb{E} \left( \sum_{k=1}^{n} \theta_k \frac{S_T(t_k)}{T^{1/2+d}} \right)^2 = \sigma^2 \mathbb{E} \left( \sum_{k=1}^{n} \theta_k B(t_k) \right)^2. \]  
(A.7)

Since \( X_k^T \) is a stationary series the relation (A.7) follows from
\[ \mathbb{E} \left( \frac{S_T(t)}{T^{1/2+d}} \right)^2 \to \sigma^2 t, \]  
(A.8)
as \( T \to \infty \).

Write
\[ S_T(t) = \sum_{k=1}^{[Tt]} (\mu_{s_k} - \mathbb{E} \mu_{s_k}) + \sum_{k=1}^{[Tt]} \epsilon_k =: S_{1,T}(t) + S_{2,T}(t). \]  
(A.9)

Then, by the independence of \( s^T \) and \( \{\epsilon_k\} \),
\[ \mathbb{E} \left( \frac{S_{1,T}(t)}{T^{1/2+d}} \right)^2 = \mathbb{E} \left( \frac{S_{1,T}(t)}{T^{1/2+d}} \right)^2 + \mathbb{E} \left( \frac{S_{2,T}(t)}{T^{1/2+d}} \right)^2 \]
and
\[ \mathbb{E} \left( \frac{S_{2,T}(t)}{T^{1/2+d}} \right)^2 = \frac{\sigma^2 [Tt]}{T^{1+2d}} \to 0. \]
Hence, it is enough to show that
\[ \mathbb{E} \left( \frac{S_{1,T}(t)}{T^{1/2+d}} \right)^2 \to \sigma^2 t. \]  
(A.10)

Observe that
\[ \mathbb{E}(S_{1,T}(t))^2 = \text{Var} \left( \sum_{k=1}^{[Tt]} \mu_{s_k} \right) = \text{Var} \left( \sum_{k=1}^{[Tt]} \mu' \xi_k \right) = \mu' \sum_{k_1, k_2=1}^{[Tt]} \text{Cov}(\xi_{k_1}, \xi_{k_2}) \mu \]
\[ = \mu' \left( [Tt] \Gamma_0 + \sum_{j=1}^{[Tt]-1} ([Tt] - j)(\Gamma_j + \Gamma'_j) \right) \mu. \]

From Lemma 2.1 and by using
\[ \sum_{j=1}^{[Tt]-1} \lambda^j = \frac{\lambda^{[Tt]} - \lambda}{\lambda - 1}, \quad \sum_{j=1}^{[Tt]-1} j \lambda^j = \frac{\lambda}{(\lambda - 1)^2} (([Tt] - 1)\lambda^{[Tt]} - [Tt] \lambda^{[Tt]-1} + 1), \]  
(A.11)
we get
\[ \mathbb{E}(S_{1,T}(t))^2 = V_{00}(\mu_0 - \mu_1)^2, \]  
(A.12)
where
\[ V_{00} = \frac{(1 - p_{00})(1 - p_{11})}{(2 - p_{00} - p_{11})^2} \left( \frac{2[Tt] \lambda}{1 - \lambda} + [Tt] + \frac{2(\lambda^{[Tt]+1} - \lambda)}{(1 - \lambda)^2} \right). \]

But by (2.3) and (1.4),
\[ \lambda = p_{00} + p_{11} + 1 = 1 - \frac{c_0}{T^{\delta_0}} - \frac{c_1}{T^{\delta_1}}. \]
Hence, as $T \to \infty$,

$$\frac{1}{1 - \lambda} = \frac{1}{c_0 T^{-\delta_0} + c_1 T^{-\delta_1}} \sim \frac{T^{\min(\delta_0, \delta_1)}}{\bar{c}},$$

where $\bar{c}$ is defined in (2.7), and

$$\lambda^{[T]} = \left(1 - \frac{c_0}{T^{-\delta_0}} - \frac{c_1}{T^{-\delta_1}}\right)^{[T]} \sim \left(1 - \frac{\bar{c}}{T^{\min(\delta_0, \delta_1)}}\right)^{[T]} \sim \exp \left\{-\bar{c}[T]^{-\min(\delta_0, \delta_1)}\right\} = o(1),$$

since $\delta_0, \delta_1 < 1$. This shows that

$$V_{00} \sim \frac{(1 - p_{00})(1 - p_{11})}{(2 - p_{00} - p_{11})^2} \frac{2[T]\lambda}{1 - \lambda} \sim \frac{(1 - p_{00})(1 - p_{11})}{(2 - p_{00} - p_{11})^2} \frac{2c_0 c_1}{\bar{c}} T^{-\min(\delta_0, \delta_1)}, \quad (A.13)$$

Now,

$$\frac{(1 - p_{00})(1 - p_{11})}{(2 - p_{00} - p_{11})^2} \sim \frac{c_0 T^{-\delta_0} c_1 T^{-\delta_1}}{c^2 T^{-2\min(\delta_0, \delta_1)}} \sim \frac{c_0 c_1}{c^2} T^{-|\delta_0 - \delta_1|},$$

since $-\delta_0 - \delta_1 + 2 \min\{\delta_0, \delta_1\} = -|\delta_0 - \delta_1|$. Thus, by (1.4),

$$V_{00} \sim \frac{2c_0 c_1 [T]}{c^3} T^{-\min(\delta_0, \delta_1) - |\delta_0 - \delta_1|} = \frac{2c_0 c_1 [T]}{c^3} T^{-2d}. \quad (A.14)$$

Relations (A.12) and (A.14) now yield

$$\frac{\mathbb{E}(S_{1,T}(1))^2}{T^{1+2d}} \sim \frac{2c_0 c_1 (\mu_0 - \mu_1)^2 t}{c^3} = \sigma_t^2,$$

which establishes (A.10).

We now turn to the convergence (A.6). In view of the decomposition (A.9) and since, by the central limit theorem, as $T \to \infty$,

$$\sum_{k=1}^{n} \theta_k S_{2,T}(t_k) \to N(0, \sigma_{2,n}^2),$$

where $0 < \sigma_{2,n}^2 < \infty$, it is enough to show that

$$\sum_{k=1}^{n} \theta_k S_{1,T}(t_k) \to N(0, \sigma_n^2), \quad (A.15)$$

where $\sigma_n^2$ is defined in (A.7). Write

$$\sum_{k=1}^{n} \theta_k S_{1,T}(t_k) \to \sum_{t=1}^{T} f_t^{(T)}(s_t^T),$$

where

$$f_t^{(T)}(x) = \sum_{k=1}^{n} \theta_k \frac{1}{T^{1/2+d}} 1_{\{t \leq [T_{k}]\}} (\mu_x - \mathbb{E} \mu_{x_t^T}).$$

Recall that $s_t^T$ may be equal to 0 or 1.

We shall apply next a central limit theorem for functionals of Markov chains. Let

$$\Delta_T := \max_{x_1, x_2, A} \left| \mathbb{P}(s_{t+1}^T \in A | s_t^T = x_1) - \mathbb{P}(s_{t+1}^T \in A | s_t^T = x_2) \right| \quad (6)$$
be the so-called contraction coefficient for the Markov chain \( s^T \). Note first that
\[
\Delta_T = \max_A \left| \mathbb{P}(s^T_{t+1} \in A|s^T_t = 0) - \mathbb{P}(s^T_{t+1} \in A|s^T_t = 1) \right|
\]
since \( x_1, x_2 \in \{0, 1\} \) and the difference \( \mathbb{P}(s^T_{t+1} \in A|s^T_t = x_1) - \mathbb{P}(s^T_{t+1} \in A|s^T_t = x_2) \) is zero whenever \( x_1 = x_2 \). Note next that the difference \( \mathbb{P}(s^T_{t+1} \in A|s^T_t = 0) - \mathbb{P}(s^T_{t+1} \in A|s^T_t = 1) \) is zero for \( A = \{0, 1\} \), and equals \( p_{00} + p_{11} - 1 \) for \( A = \{0\} \) and \( A = \{1\} \). Hence,
\[
\Delta_T = p_{00} + p_{11} - 1.
\]
By (1.4),
\[
\alpha_T := 1 - \Delta_T = \frac{c_0}{T^{\delta_0}} + \frac{c_1}{T^{\delta_1}} \sim \frac{\bar{c}}{T^{\min(\delta_0, \delta_1)}},
\]
as \( T \to \infty \). Observe that
\[
\sup_{1 \leq i \leq T} \sup_x |f_i^{(T)}(x)| \leq C_T := \frac{C}{T^{1/2+d}}.
\]
By Theorem 1.1 in Sethuraman and Varadhan (2005) (see also Theorem 1 in Dobrushin (1956)), the convergence (A.15) follows if
\[
C_T^2 \alpha_T^{-3} \left( \sum_{i=1}^{T} \text{Var}(f_i^{(T)}(s^T_i)) \right)^{-1} \to 0.
\]
(A.17)
Since \( 1_{\{t \leq T_k\}} = 1 \) for large enough \( T \), note that, for large enough \( T \),
\[
\text{Var}(f_i^{(T)}(s^T_i)) = \frac{(\sum_{k=1}^{n} \theta_k)^2}{T^{1+2d}} \text{Var}(\mu_{s^T_i}) = \frac{(\sum_{k=1}^{n} \theta_k)^2}{T^{1+2d}} \mu_0^2 / \Gamma_0 \mu.
\]
In view of (A.13), this yields
\[
\text{Var}(f_i^{(T)}(s^T_i)) = \frac{(\sum_{k=1}^{n} \theta_k)^2}{T^{1+2d}} \frac{(1 - p_{00})(1 - p_{11})}{(2 - p_{00} - p_{11})^2} (\mu_0 - \mu_1)^2 \sim \frac{(\sum_{k=1}^{n} \theta_k)^2}{T^{1+2d}} \frac{c_0 c_1}{\bar{c}^2} \frac{T^{-2d-|\delta_0 - \delta_1|}}{T}.
\]
Combining this with the asymptotic form of \( \alpha_T \) given in (A.16), we conclude that the condition (A.17) holds if
\[
\frac{1}{T^{1+2d}} T^{3 \min(\delta_0, \delta_1)} T^{2d+|\delta_0 - \delta_1|} = T^{3 \min(\delta_0, \delta_1) + |\delta_0 - \delta_1| - 1} \to 0.
\]
The convergence to 0 follows from the assumption (2.4). \( \square \)

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