Multivariate long-range dependence

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(based on joint papers with S. Kechagias, UNC, and G. Didier, Tulane)
Basic questions and a few references

How is LRD$^1$ defined in the context of multivariate (vector) time series? What multivariate LRD models are (or should be) used? A number of interesting issues concerning LRD arise for multivariate series which are not present in the univariate context.

The talk is based on

- “Definitions and representations of multivariate long-range dependent time series”, preprint, 2014, jointly with S. Kechagias. In addition, several papers on multivariate fractional Brownian motions, jointly with G. Didier.

and inspired by


$^1$LRD stands for “long-range dependence”, “long-range dependent”. Another commonly used name is “long-memory”.

Vladas Pipiras (UNC) Multivariate LRD
Outline

1. Univariate LRD series
2. Definitions of multivariate (in fact, only bivariate) LRD series
   !!! Phase parameter
3. Some models and linear representations
   !!! Causal (one-sided) and non-causal (two-sided) representations
4. Some open questions
1. Univariate LRD series

A second-order stationary series \( \{ X_n \}_{n \in \mathbb{Z}} \) is LRD with LRD parameter \( d \in (0, 1/2) \) when

**Time domain:** The autocovariance function \( \gamma(n) = \text{Cov}(X_0, X_n) \) is such that, for some \( C_1 > 0 \),

\[
\gamma(n) \sim C_1 n^{2d-1}, \text{ as } n \to +\infty. \tag{1}
\]

**Spectral domain:** The spectral density function \( f(\lambda) \) is such that, for some \( C_2 > 0 \),

\[
f(\lambda) \sim C_2 \lambda^{-2d}, \text{ as } \lambda \to 0^+. \tag{2}
\]

**Linear representations:** \( X_n = \mu + \sum_{m=0}^{\infty} \psi_m \epsilon_{n-m} \), where \( \{\epsilon_n\}_{n \in \mathbb{Z}} \) is a white noise, \( \mu \) is a constant mean and the coefficient sequence \( \{\psi_m\}_{m \geq 0} \) has a power-law behavior: for \( C_3 > 0 \),

\[
\psi_m \sim C_3 m^{d-1}, \text{ as } m \to +\infty. \tag{3}
\]

\footnote{For LRD series, \( \sum |\gamma(n)| = \infty \). In contrast, when \( \sum |\gamma(n)| < \infty \), the series is referred to as short-range dependent (SRD).}
A popular example of LRD series is FARIMA(0, d, 0) series defined as

\[ X_n = (I - B)^{-d} \epsilon_n = \sum_{m=0}^{\infty} \psi_m \epsilon_{n-m} = \sum_{m=0}^{\infty} \frac{\Gamma(m + d)}{\Gamma(m + 1) \Gamma(d)} \epsilon_{n-m}, \]

where \( B \) is the backshift operator and \( \psi_m \) are the coefficients in the Taylor expansion \((1 - z)^{-d} = \sum_{m=0}^{\infty} \psi_m z^m\) satisfying \( \psi_m \sim \frac{m^{d-1}}{\Gamma(d)} \), as \( m \to +\infty \).

The spectral density of FARIMA(0, d, 0) series is

\[ f(\lambda) = \frac{\sigma^2 |\hat{\psi}(\lambda)|^2}{2\pi} = \frac{\sigma^2}{2\pi} \left| \sum_{m=0}^{\infty} \psi_m e^{-im\lambda} \right|^2 \sim \frac{\sigma^2}{2\pi} \lambda^{-2d}, \]

where

\[ \hat{\psi}(\lambda) = (1 - e^{-i\lambda})^{-d} \sim (i\lambda)^{-d} = \lambda^{-d} e^{-i\pi d/2} \]

as \( \lambda \to 0^+ \).
A **bivariate** second-order stationary time series \( \{X_n = (X_{1,n}, X_{2,n})'\} \in \mathbb{Z} \) is LRD with parameters \( d_j \in (0, 1/2), j = 1, 2 \), when\(^3\)

**Spectral domain:** The spectral density matrix function satisfies

\[
f(\lambda) = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix} \sim \begin{pmatrix} G_{11} \lambda^{-2d_1} & G_{12} \lambda^{-(d_1+d_2)} \\ G_{21} \lambda^{-(d_1+d_2)} & G_{22} \lambda^{-2d_2} \end{pmatrix}
\]

\[
= \begin{pmatrix} g_{11} \lambda^{-2d_1} & g_{12} e^{i\phi_{12}} \lambda^{-(d_1+d_2)} \\ g_{12} e^{-i\phi_{12}} \lambda^{-(d_1+d_2)} & g_{22} \lambda^{-2d_2} \end{pmatrix}, \quad \text{as} \quad \lambda \to 0^+,
\]

for Hermitian symmetric, nonnegative definite \( G = (G_{jk}(\lambda))_{j,k=1,2} \), where \( g_{jk} \in \mathbb{R} \) and phase parameter \( \phi_{12} \in (-\pi, \pi] \).

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\(^3\)Robinson (2008), Kechagias and Pipiras (2014).
2. Definitions of multivariate LRD series

Remark 1: Phase parameters are unique to LRD. In the SRD case, $f(\lambda) = (2\pi)^{-1} \sum_{n=-\infty}^{\infty} e^{-in\lambda} \gamma(n)$ and $f(0) = (2\pi)^{-1} \sum_{n=-\infty}^{\infty} \gamma(n)$ consists of real entries. Hence, $f(\lambda) \sim G (= f(0))$, as $\lambda \to 0$, where $G$ consists of real entries.

Remark 2: Under mild conditions $f_{12}(\lambda) \sim g_{12} e^{i\phi_{12}} \lambda^{-(d_1+d_2)}$ is equivalent to $\gamma_{12}(n) = \text{Cov}(X_{1,0}, X_{2,n}) = \gamma_{21}(-n)$ satisfying

$$\gamma_{12}(n) \sim R_{12} n^{(d_1+d_2)-1}, \quad \gamma_{12}(-n) = \gamma_{21}(n) \sim R_{21} n^{(d_1+d_2)-1},$$

as $n \to +\infty$, where

$$\phi_{12} = - \arctan \left\{ \frac{R_{12} - R_{21}}{R_{12} + R_{21}} \tan \left( \frac{\pi}{2} (d_1 + d_2) \right) \right\}.$$

Note that $\phi_{12} = 0$ if and only if $R_{12} = R_{21}$. This corresponds to $\gamma_{12}(n)$ being symmetric at the two tails, that is, as $n \to +\infty$ and $n \to -\infty$. 
2. Definitions of multivariate LRD series

A look at a few bivariate time series: Internet traffic trace data: The LRD and phase parameters are estimated using the bivariate extension of the local Whittle method in Robinson (2008):
A look at a few bivariate time series: Temperature data: The LRD and phase parameters are estimated using the bivariate extension of the local Whittle method in Robinson (2008).
3. Some models and linear representations

**Question:** What about models for bivariate LRD series?

**Example:** One common example is a bivariate FARIMA series defined as

\[
X_n = \begin{pmatrix} X_{1,n} \\ X_{2,n} \end{pmatrix} = \begin{pmatrix} (I - B)^{-d_1} \eta_{1,n} \\ (I - B)^{-d_2} \eta_{2,n} \end{pmatrix} = (I - B)^{-D} \eta_n = (I - B)^{-D} Q_+ \epsilon_n,
\]

with \( D = \text{diag}(d_1, d_2) \) and bivariate white noise \( \eta_n \) satisfying

\[
E \eta_n \eta_n' = \Sigma = (\sigma_{jk})_{j,k=1,2} = Q_+ Q_+'.
\]

The spectral density matrix is

\[
f(\lambda) = \frac{1}{2\pi} (1 - e^{-i\lambda})^{-D} \Sigma (1 - e^{i\lambda})^{-D} \sim \frac{1}{2\pi} \lambda^{-D} e^{-i \frac{\pi D}{2} } \Sigma e^{i \frac{\pi D}{2} } \lambda^{-D}
\]

\[
= \frac{1}{2\pi} \begin{pmatrix} \sigma_{11} \lambda^{-2d_1} & \sigma_{12} e^{-i \frac{\pi}{2} (d_1 - d_2)} \lambda^{-(d_1 + d_2)} \\ \sigma_{12} e^{i \frac{\pi}{2} (d_1 - d_2)} \lambda^{-(d_1 + d_2)} & \sigma_{22} \lambda^{-2d_2} \end{pmatrix}.
\]

Thus, \( \phi_{12} = -\frac{\pi}{2} (d_1 - d_2) \).
Result 1: If $A^+ = (\alpha_{jk}^+)_{j,k=1,2}$ is a real matrix and 
\{\psi_m = (\psi_{jk,m})_{j,k=1,2}\}_{m \geq 0}$ is a sequence of real matrices such that 
\[ \psi_{jk,m} \sim \alpha_{jk}^+ m^{d_j-1}, \quad \text{as} \quad m \to +\infty, \]
then a causal (one-sided) linear time series $X_n = \sum_{m=0}^{\infty} \Psi_m \epsilon_{n-m}$ with $E\epsilon_n\epsilon'_n = I$ is multivariate LRD in the spectral domain with
\[ G_{jk} = \frac{\Gamma(d_j)\Gamma(d_k)}{2\pi} (A^+ (A^+)^*)_{jk} e^{-i\frac{\pi}{2} (d_j-d_k)} \]
and hence the phase parameter
\[ \phi_{12} = -\frac{\pi}{2} (d_1 - d_2). \]

Question: What about more general phases? For that matter, what about zero phases (associated with symmetry)? (Note that $\phi_{12} = 0$ above iff $d_1 = d_2$. )
3. Some models and linear representations

**Result 2:** If $A^- = (\alpha^-_{jk})_{j,k=1,2}$, $A^+ = (\alpha^+_{jk})_{j,k=1,2}$ are real matrices and 
$
\{\Psi_m = (\psi_{jk,m})_{j,k=1,2}\}_{m \in \mathbb{Z}}
$
is a sequence of real matrices such that 
\[
\psi_{jk,m} \sim \alpha^+_{jk} m^{d_j-1}, \quad \text{as } m \to +\infty,
\]
\[
\psi_{jk,m} \sim \alpha^-_{jk} (-m)^{d_j-1}, \quad \text{as } m \to -\infty,
\]
then a non-causal (two-sided) linear time series $X_n = \sum_{m=-\infty}^{\infty} \Psi_m \epsilon_{n-m}$ with $E\epsilon_n \epsilon'_n = I$ is multivariate LRD in the spectral domain with 
\[
G_{jk} = \frac{1}{2\pi} \left( (FA^+ + F^* A^-)(FA^+ + F^* A^-)^* \right)_{jk},
\]
where $F = \text{diag}(\Gamma(d_1) e^{-i \frac{\pi}{2} d_1}, \Gamma(d_2) e^{-i \frac{\pi}{2} d_2})$. This leads to a general phase parameter.
Example: bivariate FARIMA$(0, D, 0)$ series with general phase: Let $D = \text{diag}(d_1, d_2)$ with $d_j < 1/2$, $j = 1, 2$, and $Q_+ = (q_{jk}^+)$, $Q_- = (q_{jk}^-)$ be real matrices. Let also $\{\epsilon_n\}_{n \in \mathbb{Z}}$ be white noise series satisfying $E\epsilon_n\epsilon_n' = I$. Define a bivariate FARIMA$(0, D, 0)$ series as

$$X_n = (I - B)^{-D}Q_+\epsilon_n + (I - B^{-1})^{-D}Q_-\epsilon_n.$$ 

The series $X_n$ is given by a non-causal linear representation in general.

**Result 3:** The autocovariance matrix function of FARIMA$(0, D, 0)$ series can be computed explicitly.\(^4\)

**Question:** What about causal linear representations with general phases?\(^5\)

For that matter, what about zero phase (associated with symmetry)?

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\(^4\)Kechagias and Pipiras (2014).

\(^5\)Causal representations should exist by the Wiener-Paley theorem.
3. Some models and linear representations

**Case of zero phase:** We are looking for real $\Psi_m$ such that, as $\lambda \to 0^+$,

$$
\left( \sum_{m=0}^{\infty} \psi_m e^{-im\lambda} \right) \left( \sum_{m=0}^{\infty} \psi_m^* e^{im\lambda} \right) \sim \left( G_{jk} \lambda^{-(d_j+d_k)} \right)_{j,k=1,2},
$$

with real $G_{jk}$. This relation would follow from

$$
\left( \sum_{m=0}^{\infty} \psi_{jk,m} e^{-im\lambda} \right) \left( \sum_{m=0}^{\infty} \psi_{j'k',m} e^{im\lambda} \right) \sim c_{jk,j'k'} \lambda^{-(d_j+d_{j'})}
$$

with real $c_{jk,j'k'}$. We cannot choose real coefficients $\psi_m$ such that

$$
\sum_{m=0}^{\infty} \psi_m e^{-im\lambda} \sim c \lambda^{-d}
$$

with real $c$. What about

$$
\sum_{m=0}^{\infty} \psi_m e^{-im\lambda} \sim c \lambda^{-d} e^{ih(\lambda)},
$$

where $h(\lambda) \to \infty$ and $h(\lambda)$ does not depend on $d$?
3. Some models and linear representations

**Trigonometric power-law coefficients:** For $m \geq 0$ (and $a \in (0, 1)$), set

$$c_m^{a,d_0} = m^{d_0-1} \cos(2\pi m^a), \quad s_m^{a,d_0} = m^{d_0-1} \sin(2\pi m^a).$$

**Log-log plot:**

![Log-log plot](image)
3. Some models and linear representations

\[ c_m^{a,d_0} = m^{d_0-1} \cos(2\pi m^a), \quad s_m^{a,d_0} = m^{d_0-1} \sin(2\pi m^a). \]

**Result 4:** If \( a \in (0, 1) \) and \( d_0 \in (\frac{a}{2}, \frac{1}{2}) \), then as \( \lambda \to 0^+ \),

\[
\sum_{m=0}^{\infty} c_m^{a,d_0} e^{-im\lambda} \sim c_{a,d_0} \lambda^{-d} e^{ih(\lambda)}, \quad \sum_{m=0}^{\infty} s_m^{a,d_0} e^{-im\lambda} \sim ic_{a,d_0} \lambda^{-d} e^{ih(\lambda)}
\]

with \( h(\lambda) = \xi_a \lambda^{-a/(1-a)} - \frac{\pi}{4} \) and known real \( c_{a,d_0} \) and \( \xi_a \), and

\[
d = \frac{d_0 - \frac{a}{2}}{1-a} < d_0.
\]

**Question:** Why not “continuous” at \( a = 0 \)?

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\(^{6}\) Adapting and refining the work of Wainger (1965), *Special trigonometric series in k-dimensions*, Memoirs of the American Mathematical Society (59).
3. Some models and linear representations

**Result 5:** If \( A^+ = (\alpha_{jk}^+)_{j,k=1,2} \) is a real matrix and \( \{\Psi_m = (\psi_{jk,m})_{j,k=1,2}\}_{m \geq 0} \) is a sequence of real matrices with

\[
\psi_{jk,m} = \alpha_{jk}^+ \cos(2\pi m a) m^{d_{0,j} - 1},
\]

where \( a \in (0, 1) \) and \( d_{0,j} \in (a/2, 1/2) \), then a causal linear time series \( X_n = \sum_{m=0}^{\infty} \Psi_m \epsilon_{n-m} \) is multivariate LRD in the spectral domain with

\[
d_j = \frac{d_{0,j} - a}{2}, \quad G_{jk} = c(A^+(A^+)^*)_{jk}
\]

(\( c \) is a constant) and hence the phase parameter

\[
\phi_{12} = 0.
\]

**Result 6:** Causal multivariate LRD series with more general phase can be constructed through the coefficients which are linear combinations of the trigonometric power-law coefficients.
4. Some open questions

- “Robustness” of trigonometric power-law coefficients; inverse filter
- Estimation issues in linear time series with trigonometric power-law coefficients
- A (FARIMA-like) family of causal bivariate LRD series with general phase that is suitable in modeling
- Causal representations of bivariate fractional Brownian motions
- What difference does the phase make in modeling?

Thank You!  Questions?